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$C(21.5) = \frac{V}{15} - \frac{V}{900}(21.5) + \frac{V}{38{,}700}(21.5)^2 \approx 0.05472V$, $C(0) = \frac{V}{15} \approx 0.06667V$, and

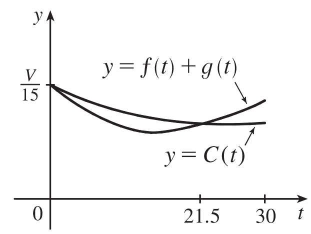
$C(30) = \frac{V}{15} - \frac{V}{900}(30) + \frac{V}{38{,}700}(30)^2 \approx 0.05659V$, so the absolute minimum is $C(21.5) \approx 0.05472V$.

(d) As in part (c), we have $C(t) = \frac{V}{15} - \frac{V}{900}t + \frac{V}{38{,}700}t^2$, so $C(t) = f(t) + g(t) \quad \Leftrightarrow$

$\frac{V}{15} - \frac{V}{900}t + \frac{V}{38{,}700}t^2 = \frac{V}{15} - \frac{V}{450}t + \frac{V}{12{,}900}t^2 \quad \Leftrightarrow$

$t^2\left(\frac{1}{12{,}900} - \frac{1}{38{,}700}\right) = t\left(\frac{1}{450} - \frac{1}{900}\right) \quad \Leftrightarrow \quad t = \frac{1/900}{2/38{,}700} = \frac{43}{2} = 21.5.$

This is the value of t that we obtained as the critical number of C in part (c), so we have verified the result of (a) in this case.



33. (a) Let $F(t) = \int_0^t f(s)\,ds$. Then, by FTC1, $F'(t) = f(t) =$ rate of depreciation, so $F(t)$ represents the loss in value over the interval $[0, t]$.

(b) $C(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right] = \frac{A + F(t)}{t}$ represents the average expenditure per unit of t during the interval $[0, t]$, assuming that there has been only one overhaul during that time period. The company wants to minimize average expenditure.

(c) $C(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right]$. Using FTC1, we have $C'(t) = -\frac{1}{t^2}\left[A + \int_0^t f(s)\,ds\right] + \frac{1}{t}f(t)$.

$C'(t) = 0 \quad \Rightarrow \quad t\,f(t) = A + \int_0^t f(s)\,ds \quad \Rightarrow \quad f(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right] = C(t).$

5.5 The Substitution Rule

1. Let $u = -x$. Then $du = -\,dx$, so $dx = -\,du$. Thus, $\int e^{-x}dx = \int e^u(-du) = -e^u + C = -e^{-x} + C$. Don't forget that it is often very easy to check an indefinite integration by differentiating your answer. In this case,

$\frac{d}{dx}(-e^{-x} + C) = -[e^{-x}(-1)] = e^{-x}$, the desired result.

2. Let $u = 2 + x^4$. Then $du = 4x^3\,dx$ and $x^3\,dx = \frac{1}{4}\,du$,

so $\int x^3(2 + x^4)^5\,dx = \int u^5\left(\frac{1}{4}\,du\right) = \frac{1}{4}\frac{u^6}{6} + C = \frac{1}{24}(2 + x^4)^6 + C.$

3. Let $u = x^3 + 1$. Then $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}\,du$, so

$$\int x^2\sqrt{x^3 + 1}\,dx = \int \sqrt{u}\left(\tfrac{1}{3}\,du\right) = \frac{1}{3}\frac{u^{3/2}}{3/2} + C = \frac{1}{3}\cdot\frac{2}{3}u^{3/2} + C = \tfrac{2}{9}(x^3 + 1)^{3/2} + C.$$

4. Let $u = 1 - 6t$. Then $du = -6\,dt$ and $dt = -\frac{1}{6}\,du$, so

$$\int \frac{dt}{(1 - 6t)^4} = \int \frac{-\frac{1}{6}\,du}{u^4} = -\tfrac{1}{6}\int u^{-4}\,du = -\frac{1}{6}\frac{u^{-3}}{-3} + C = \frac{1}{18u^3} + C = \frac{1}{18(1 - 6t)^3} + C.$$

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5. Let $u = \cos\theta$. Then $du = -\sin\theta\,d\theta$ and $\sin\theta\,d\theta = -du$, so

$$\int \cos^3\theta\,\sin\theta\,d\theta = \int u^3\,(-du) = -\frac{u^4}{4} + C = -\tfrac{1}{4}\cos^4\theta + C.$$

6. Let $u = 1/x$. Then $du = -1/x^2\,dx$ and $1/x^2\,dx = -du$, so

$$\int \frac{\sec^2(1/x)}{x^2}\,dx = \int \sec^2 u\,(-du) = -\tan u + C = -\tan(1/x) + C.$$

7. Let $u = x^2$. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, so $\int x\sin(x^2)\,dx = \int \sin u\left(\frac{1}{2}\,du\right) = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos(x^2) + C$.

8. Let $u = x^3 + 5$. Then $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}\,du$, so

$\int x^2(x^3+5)^9\,dx = \int u^9\left(\frac{1}{3}\,du\right) = \frac{1}{3}\cdot\frac{1}{10}u^{10} + C = \frac{1}{30}(x^3+5)^{10} + C.$

9. Let $u = 3x - 2$. Then $du = 3\,dx$ and $dx = \frac{1}{3}\,du$, so $\int (3x-2)^{20}\,dx = \int u^{20}\left(\frac{1}{3}\,du\right) = \frac{1}{3}\cdot\frac{1}{21}u^{21} + C = \frac{1}{63}(3x-2)^{21} + C$.

10. Let $u = 3t + 2$. Then $du = 3\,dt$ and $dt = \frac{1}{3}\,du$, so

$$\int (3t+2)^{2.4}\,dt = \int u^{2.4}\left(\tfrac{1}{3}\,du\right) = \frac{1}{3}\,\frac{u^{3.4}}{3.4} + C = \frac{1}{10.2}(3t+2)^{3.4} + C.$$

11. Let $u = \pi t$. Then $du = \pi\,dt$ and $dt = \frac{1}{\pi}\,du$, so $\int \sin\pi t\,dt = \int \sin u\left(\frac{1}{\pi}\,du\right) = \frac{1}{\pi}(-\cos u) + C = -\frac{1}{\pi}\cos\pi t + C$.

12. Let $u = e^x$. Then $du = e^x\,dx$, so $\int e^x\cos(e^x)\,dx = \int \cos u\,du = \sin u + C = \sin(e^x) + C$.

13. Let $u = \ln x$. Then $du = \dfrac{dx}{x}$, so $\displaystyle\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}(\ln x)^3 + C$.

14. Let $u = x^2 + 1$. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, so

$$\int \frac{x}{(x^2+1)^2}\,dx = \int u^{-2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\cdot\frac{-1}{u} + C = \frac{-1}{2u} + C = \frac{-1}{2(x^2+1)} + C.$$

15. Let $u = 5 - 3x$. Then $du = -3\,dx$ and $dx = -\frac{1}{3}\,du$, so

$$\int \frac{dx}{5-3x} = \int \frac{1}{u}\left(-\tfrac{1}{3}\,du\right) = -\tfrac{1}{3}\ln|u| + C = -\tfrac{1}{3}\ln|5-3x| + C.$$

16. Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\,dx$ and $2\,du = \dfrac{1}{\sqrt{x}}\,dx$, so

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \sin u\,(2\,du) = -2\cos u + C = -2\cos\sqrt{x} + C.$$

17. Let $u = 3ax + bx^3$. Then $du = (3a + 3bx^2)\,dx = 3(a + bx^2)\,dx$, so

$$\int \frac{a+bx^2}{\sqrt{3ax+bx^3}}\,dx = \int \frac{\frac{1}{3}\,du}{u^{1/2}} = \frac{1}{3}\int u^{-1/2}\,du = \tfrac{1}{3}\cdot 2u^{2} + C = \tfrac{2}{3}\sqrt{3ax+bx^3} + C.$$

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18. Let $u = z^3 + 1$. Then $du = 3z^2\,dz$ and $\frac{1}{3}\,du = z^2\,dz$, so

$$\int \frac{z^2}{z^3+1}\,dz = \int \frac{1}{u}\left(\tfrac{1}{3}\,du\right) = \tfrac{1}{3}\ln|u| + C = \tfrac{1}{3}\ln|z^3+1| + C$$

19. Let $u = 1 + e^x$. Then $du = e^x\,dx$, so $\int e^x\sqrt{1+e^x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C$.

Or: Let $u = \sqrt{1+e^x}$. Then $u^2 = 1 + e^x$ and $2u\,du = e^x\,dx$, so

$\int e^x\sqrt{1+e^x}\,dx = \int u \cdot 2u\,du = \frac{2}{3}u^3 + C = \frac{2}{3}(1+e^x)^{3/2} + C$.

20. Let $u = 2\theta$. Then $du = 2\,d\theta$ and $d\theta = \frac{1}{2}\,du$, so

newline $\int \sec 2\theta\,\tan 2\theta\,d\theta = \int \sec u\tan u\left(\frac{1}{2}\,du\right) = \frac{1}{2}\sec u + C = \frac{1}{2}\sec 2\theta + C$.

21. Let $u = \sin x$. Then $du = \cos x\,dx$, so $\displaystyle\int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{1}{u^2}\,du = \int u^{-2}\,du = \frac{u^{-1}}{-1} + C = -\frac{1}{u} + C = -\frac{1}{\sin x} + C$

[or $-\csc x + C$].

22. Let $u = \tan^{-1}x$. Then $du = \dfrac{dx}{1+x^2}$, so $\displaystyle\int \frac{\tan^{-1}x}{1+x^2}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{(\tan^{-1}x)^2}{2} + C$.

23. Let $u = x^3 + 3x$. Then $du = (3x^2+3)\,dx$ and $\frac{1}{3}\,du = (x^2+1)\,dx$, so

$\int (x^2+1)(x^3+3x)^4\,dx = \int u^4\left(\frac{1}{3}\,du\right) = \frac{1}{3}\cdot\frac{1}{5}u^5 + C = \frac{1}{15}(x^3+3x)^5 + C$.

24. Let $u = \ln x$. Then $du = (1/x)\,dx$, so $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx = \int \sin u\,du = -\cos u + C = -\cos(\ln x) + C$.

25. Let $u = \cot x$. Then $du = -\csc^2 x\,dx$ and $\csc^2 x\,dx = -du$, so

$\displaystyle\int \sqrt{\cot x}\,\csc^2 x\,dx = \int \sqrt{u}\,(-du) = -\frac{u^{3/2}}{3/2} + C = -\tfrac{2}{3}(\cot x)^{3/2} + C$.

26. Let $u = \dfrac{\pi}{x}$. Then $du = -\dfrac{\pi}{x^2}\,dx$ and $\dfrac{1}{x^2}\,dx = -\dfrac{1}{\pi}\,du$, so

$$\int \frac{\cos(\pi/x)}{x^2}\,dx = \int \cos u\left(-\frac{1}{\pi}\,du\right) = -\frac{1}{\pi}\sin u + C = -\frac{1}{\pi}\sin\frac{\pi}{x} + C$$

27. Let $u = \sin^{-1}x$. Then $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$, so $\displaystyle\int \frac{dx}{\sqrt{1-x^2}\,\sin^{-1}x} = \int \frac{1}{u}\,du = \ln|u| + C = \ln\left|\sin^{-1}x\right| + C$.

28. Let $u = 1 + \tan t$. Then $du = \sec^2 t\,dt$, so

$\displaystyle\int \frac{dt}{\cos^2 t\,\sqrt{1+\tan t}} = \int \frac{\sec^2 t\,dt}{\sqrt{1+\tan t}} = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{1+\tan t} + C$.

29. Let $u = \sec x$. Then $du = \sec x\,\tan x\,dx$, so

$\int \sec^3 x\,\tan x\,dx = \int \sec^2 x\,(\sec x\tan x)\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\sec^3 x + C$.

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30. Let $u = 2 + x$. Then $du = dx$, $x = u - 2$, and $x^2 = (u-2)^2$, so

$$\begin{aligned}\int x^2\sqrt{2+x}\,dx &= \int (u-2)^2\sqrt{u}\,du = \int (u^2 - 4u + 4)u^{1/2}\,du = \int (u^{5/2} - 4u^{3/2} + 4u^{1/2})\,du \\ &= \tfrac{2}{7}u^{7/2} - \tfrac{8}{5}u^{5/2} + \tfrac{8}{3}u^{3/2} + C = \tfrac{2}{7}(2+x)^{7/2} - \tfrac{8}{5}(2+x)^{5/2} + \tfrac{8}{3}(2+x)^{3/2} + C\end{aligned}$$

31. Let $u = 2x + 5$. Then $du = 2\,dx$ and $x = \frac{1}{2}(u-5)$, so

$$\begin{aligned}\int x(2x+5)^8\,dx &= \int \tfrac{1}{2}(u-5)u^8\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{4}\int (u^9 - 5u^8)\,du \\ &= \tfrac{1}{4}\left(\tfrac{1}{10}u^{10} - \tfrac{5}{9}u^9\right) + C = \tfrac{1}{40}(2x+5)^{10} - \tfrac{5}{36}(2x+5)^9 + C\end{aligned}$$

32. Let $u = e^x + 1$. Then $du = e^x\,dx$, so $\displaystyle\int \frac{e^x}{e^x+1}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln(e^x + 1) + C$.

33. $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx = 2\int \frac{\sin x\cos x}{1+\cos^2 x}\,dx = 2I$. Let $u = \cos x$. Then $du = -\sin x\,dx$, so

$$2I = -2\int \frac{u\,du}{1+u^2} = -2\cdot\tfrac{1}{2}\ln(1+u^2) + C = -\ln(1+u^2) + C = -\ln(1+\cos^2 x) + C.$$

Or: Let $u = 1 + \cos^2 x$.

34. Let $u = \cos x$. Then $du = -\sin x\,dx$ and $\sin x\,dx = -du$, so

$$\int \frac{\sin x}{1+\cos^2 x}\,dx = \int \frac{-du}{1+u^2} = -\tan^{-1} u + C = -\tan^{-1}(\cos x) + C.$$

35. Let $u = 1 + x^2$. Then $du = 2x\,dx$, so

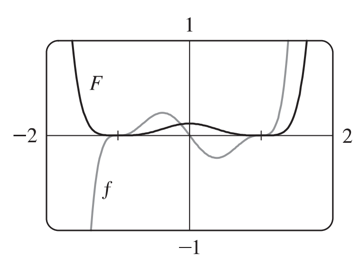
$$\begin{aligned}\int \frac{1+x}{1+x^2}\,dx &= \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx = \tan^{-1} x + \int \frac{\frac{1}{2}\,du}{u} = \tan^{-1} x + \tfrac{1}{2}\ln|u| + C \\ &= \tan^{-1} x + \tfrac{1}{2}\ln\left|1+x^2\right| + C = \tan^{-1} x + \tfrac{1}{2}\ln\left(1+x^2\right) + C \quad [\text{since } 1+x^2 > 0].\end{aligned}$$

36. Let $u = x^2$. Then $du = 2x\,dx$, so $\displaystyle\int \frac{x}{1+x^4}\,dx = \int \frac{\frac{1}{2}\,du}{1+u^2} = \tfrac{1}{2}\tan^{-1} u + C = \tfrac{1}{2}\tan^{-1}(x^2) + C$.

37. $f(x) = x(x^2-1)^3$. $\quad u = x^2 - 1 \;\Rightarrow\; du = 2x\,dx$, so

$$\int x(x^2-1)^3\,dx = \int u^3\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{8}u^4 + C = \tfrac{1}{8}(x^2-1)^4 + C$$

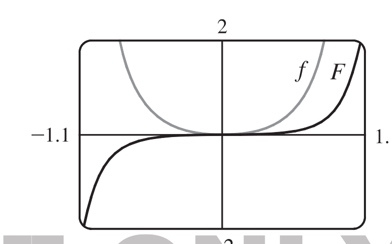
Where f is positive (negative), F is increasing (decreasing). Where f changes from negative to positive (positive to negative), F has a local minimum (maximum).



38. $f(\theta) = \tan^2\theta\,\sec^2\theta$. $u = \tan\theta \;\Rightarrow\; du = \sec^2\theta\,d\theta$, so

$$\int \tan^2\theta\,\sec^2\theta\,d\theta = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\tan^3\theta + C$$

Note that f is positive and F is increasing. At $x = 0$, $f = 0$ and F has a horizontal tangent.



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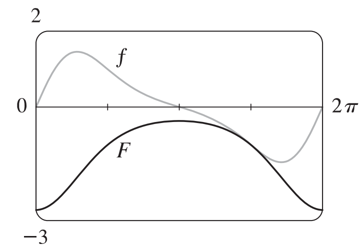
39. $f(x) = e^{\cos x} \sin x$. $u = \cos x \quad \Rightarrow \quad du = -\sin x\,dx$, so

$$\int e^u\,(-du) = -e^u + C = -e^{\cos x} + C$$

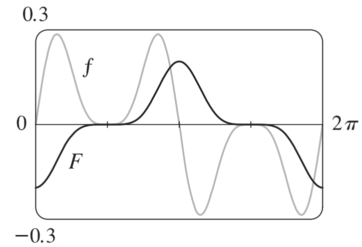
Note that at $x = \pi$, f changes from positive to negative and F has a local maximum. Also, both f and F are periodic with period 2π, so at $x = 0$ and at $x = 2\pi$, f changes from negative to positive and F has a local minimum.



40. $f(x) = \sin x \cos^4 x$. $u = \cos x \quad \Rightarrow \quad du = -\sin x\,dx$, so

$$\int \sin x \cos^4 x\,dx = \int u^4\,(-du) = -\tfrac{1}{5}u^5 + C = -\tfrac{1}{5}\cos^5 x + C$$

Note that at $x = \pi$, f changes from positive to negative and F has a local maximum. Also, both f and F are periodic with period 2π, so at $x = 0$ and at $x = 2\pi$, f changes from negative to positive and F has a local minimum.



41. Let $u = \frac{\pi}{2}t$, so $du = \frac{\pi}{2}\,dt$. When $t = 0$, $u = 0$; when $t = 1$, $u = \frac{\pi}{2}$. Thus,

$$\int_0^1 \cos(\pi t/2)\,dt = \int_0^{\pi/2} \cos u\,\left(\tfrac{2}{\pi}\,du\right) = \tfrac{2}{\pi}\left[\sin u\right]_0^{\pi/2} = \tfrac{2}{\pi}\left(\sin\tfrac{\pi}{2} - \sin 0\right) = \tfrac{2}{\pi}(1-0) = \tfrac{2}{\pi}$$

42. Let $u = 3t - 1$, so $du = 3\,dt$. When $t = 0$, $u = -1$; when $t = 1$, $u = 2$. Thus,

$$\int_0^1 (3t-1)^{50}\,dt = \int_{-1}^2 u^{50}\,\left(\tfrac{1}{3}\,du\right) = \tfrac{1}{3}\left[\tfrac{1}{51}u^{51}\right]_{-1}^2 = \tfrac{1}{153}\left[2^{51} - (-1)^{51}\right] = \tfrac{1}{153}(2^{51}+1)$$

43. Let $u = 1 + 7x$, so $du = 7\,dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = 8$. Thus,

$$\int_0^1 \sqrt[3]{1+7x}\,dx = \int_1^8 u^{1/3}\left(\tfrac{1}{7}\,du\right) = \tfrac{1}{7}\left[\tfrac{3}{4}u^{4/3}\right]_1^8 = \tfrac{3}{28}(8^{4/3} - 1^{4/3}) = \tfrac{3}{28}(16-1) = \tfrac{45}{28}$$

44. Let $u = x^2$, so $du = 2x\,dx$. When $x = 0$, $u = 0$; when $x = \sqrt{\pi}$, $u = \pi$. Thus,

$$\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx = \int_0^\pi \cos u\,\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\left[\sin u\right]_0^\pi = \tfrac{1}{2}(\sin\pi - \sin 0) = \tfrac{1}{2}(0-0) = 0.$$

45. Let $u = 1 + 2x^3$, so $du = 6x^2\,dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = 3$. Thus,

$$\int_0^1 x^2\left(1+2x^3\right)^5\,dx = \int_1^3 u^5\left(\tfrac{1}{6}\,du\right) = \tfrac{1}{6}\left[\tfrac{1}{6}u^6\right]_1^3 = \tfrac{1}{36}(3^6 - 1^6) = \tfrac{1}{36}(729-1) = \tfrac{728}{36} = \tfrac{182}{9}.$$

46. Let $u = \pi t$, so $du = \pi\,dt$. When $t = \frac{1}{6}$, $u = \frac{\pi}{6}$; when $t = \frac{1}{2}$, $u = \frac{\pi}{2}$. Thus,

$$\int_{1/6}^{1/2} \csc\pi t\,\cot\pi t\,dt = \int_{\pi/6}^{\pi/2} \csc u\,\cot u\,\left(\tfrac{1}{\pi}\,du\right) = \tfrac{1}{\pi}\left[-\csc u\right]_{\pi/6}^{\pi/2} = -\tfrac{1}{\pi}(1-2) = \tfrac{1}{\pi}.$$

47. Let $u = \sqrt{x}$, so $du = \dfrac{1}{2\sqrt{x}}\,dx$. When $x = 1$, $u = 1$; when $x = 4$, $u = 2$.

Thus, $\displaystyle\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_1^2 e^u\,(2\,du) = 2\left[e^u\right]_1^2 = 2(e^2 - e).$

48. Let $u = \sin x$, so $du = \cos x\,dx$. When $x = 0$, $u = 0$; when $x = \frac{\pi}{2}$, $u = 1$. Thus,

$$\int_0^{\pi/2} \cos x\,\sin(\sin x)\,dx = \int_0^1 \sin u\,du = \left[-\cos u\right]_0^1 = -(\cos 1 - 1) = 1 - \cos 1.$$

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49. $\int_{-\pi/4}^{\pi/4}(x^3 + x^4 \tan x)\,dx = 0$ by Theorem 6(b), since $f(x) = x^3 + x^4 \tan x$ is an odd function.

50. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1 + x^6}\,dx = 0$ by Theorem 6(b), since $f(x) = \dfrac{x^2 \sin x}{1 + x^6}$ is an odd function.

51. Let $u = x - 1$, so $u + 1 = x$ and $du = dx$. When $x = 1$, $u = 0$; when $x = 2$, $u = 1$. Thus,

$$\int_1^2 x\sqrt{x-1}\,dx = \int_0^1 (u+1)\sqrt{u}\,du = \int_0^1 (u^{3/2} + u^{1/2})\,du = \left[\tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right]_0^1 = \tfrac{2}{5} + \tfrac{2}{3} = \tfrac{16}{15}.$$

52. Assume $a > 0$. Let $u = a^2 - x^2$, so $du = -2x\,dx$. When $x = 0$, $u = a^2$; when $x = a$, $u = 0$. Thus,

$$\int_0^a x\sqrt{a^2 - x^2}\,dx = \int_{a^2}^0 u^{1/2}\left(-\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int_0^{a^2} u^{1/2}\,du = \tfrac{1}{2}\cdot\left[\tfrac{2}{3}u^{3/2}\right]_0^{a^2} = \tfrac{1}{3}a^3.$$

53. Let $u = e^z + z$, so $du = (e^z + 1)\,dz$. When $z = 0$, $u = 1$; when $z = 1$, $u = e + 1$. Thus,

$$\int_0^1 \frac{e^z + 1}{e^z + z}\,dz = \int_1^{e+1} \frac{1}{u}\,du = \Big[\ln|u|\Big]_1^{e+1} = \ln|e+1| - \ln|1| = \ln(e+1).$$

54. Let $u = \sin^{-1} x$, so $du = \dfrac{dx}{\sqrt{1-x^2}}$. When $x = 0$, $u = 0$; when $x = \frac{1}{2}$, $u = \frac{\pi}{6}$. Thus,

$$\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/6} u\,du = \left[\frac{u^2}{2}\right]_0^{\pi/6} = \frac{\pi^2}{72}.$$

55. Let $u = \ln x$, so $du = \dfrac{dx}{x}$. When $x = e$, $u = 1$; when $x = e^4$; $u = 4$. Thus,

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_1^4 u^{-1/2}\,du = 2\Big[u^{1/2}\Big]_1^4 = 2(2-1) = 2.$$

56. Let $u = \dfrac{2\pi t}{T} - \alpha$, so $du = \dfrac{2\pi}{T}dt$. When $t = 0$, $u = -\alpha$; when $t = \dfrac{T}{2}$, $u = \pi - \alpha$. Thus,

$$\begin{aligned}\int_0^{\pi/2} \sin\left(\frac{2\pi t}{T} - \alpha\right)dt &= \int_{-\alpha}^{\pi-\alpha} \sin u\left(\frac{T}{2\pi}\,du\right) = \frac{T}{2\pi}\left[-\cos u\right]_{-\alpha}^{\pi-\alpha} = -\frac{T}{2\pi}[\cos(\pi-\alpha) - \cos(-\alpha)]\\ &= -\frac{T}{2\pi}(-\cos\alpha - \cos\alpha) = -\frac{T}{2\pi}(-2\cos\alpha) = \frac{T}{\pi}\cos\alpha\end{aligned}$$

57. Let $u = 1 + \sqrt{x}$, so $du = \dfrac{1}{2\sqrt{x}}\,dx \;\Rightarrow\; 2\sqrt{x}\,du = dx \;\Rightarrow\; 2(u-1)\,du = dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = 2$. Thus,

$$\begin{aligned}\int_0^1 \frac{dx}{(1+\sqrt{x})^4} &= \int_1^2 \frac{1}{u^4}\cdot[2(u-1)\,du] = 2\int_1^2\left(\frac{1}{u^3} - \frac{1}{u^4}\right)du = 2\left[-\frac{1}{2u^2} + \frac{1}{3u^3}\right]_1^2\\ &= 2\left[\left(-\tfrac{1}{8} + \tfrac{1}{24}\right) - \left(-\tfrac{1}{2} + \tfrac{1}{3}\right)\right] = 2\left(\tfrac{1}{12}\right) = \tfrac{1}{6}\end{aligned}$$

58. If $f(x) = \sin\sqrt[3]{x}$, then $f(-x) = \sin\sqrt[3]{-x} = \sin(-\sqrt[3]{x}) = -\sin\sqrt[3]{x} = -f(x)$, so f is an odd function. Now $I = \int_{-2}^3 \sin\sqrt[3]{x}\,dx = \int_{-2}^2 \sin\sqrt[3]{x}\,dx + \int_2^3 \sin\sqrt[3]{x}\,dx = I_1 + I_2$. $I_1 = 0$ by Theorem 6(b). To estimate I_2, note that

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$2 \le x \le 3 \;\Rightarrow\; \sqrt[3]{2} \le \sqrt[3]{x} \le \sqrt[3]{3}$ [≈ 1.44] $\Rightarrow\; 0 \le \sqrt[3]{x} \le \frac{\pi}{2}$ [≈ 1.57] $\Rightarrow\; \sin 0 \le \sin \sqrt[3]{x} \le \sin \frac{\pi}{2}$ [since sine is increasing on this interval] $\Rightarrow\; 0 \le \sin \sqrt[3]{x} \le 1$. By comparison property 8, $0(3-2) \le I_2 \le 1(3-2) \;\Rightarrow$ $0 \le I_2 \le 1 \;\Rightarrow\; 0 \le I \le 1$.

59. First write the integral as a sum of two integrals:

$I = \int_{-2}^{2}(x+3)\sqrt{4-x^2}\,dx = I_1 + I_2 = \int_{-2}^{2} x\sqrt{4-x^2}\,dx + \int_{-2}^{2} 3\sqrt{4-x^2}\,dx$. $I_1 = 0$ by Theorem 6(b), since $f(x) = x\sqrt{4-x^2}$ is an odd function and we are integrating from $x=-2$ to $x=2$. We interpret I_2 as three times the area of a semicircle with radius 2, so $I = 0 + 3\cdot\frac{1}{2}\left(\pi\cdot 2^2\right) = 6\pi$.

60. Let $u = x^2$. Then $du = 2x\,dx$ and the limits are unchanged ($0^2 = 0$ and $1^2 = 1$), so

$I = \int_0^1 x\sqrt{1-x^4}\,dx = \frac{1}{2}\int_0^1 \sqrt{1-u^2}\,du$. But this integral can be interpreted as the area of a quarter-circle with radius 1.

So $I = \frac{1}{2}\cdot\frac{1}{4}\left(\pi\cdot 1^2\right) = \frac{1}{8}\pi$.

61. First Figure Let $u = \sqrt{x}$, so $x = u^2$ and $dx = 2u\,du$. When $x=0$, $u=0$; when $x=1$, $u=1$. Thus,

$$A_1 = \int_0^1 e^{\sqrt{x}}\,dx = \int_0^1 e^u(2u\,du) = 2\int_0^1 ue^u\,du.$$

Second Figure $A_2 = \int_0^1 2xe^x\,dx = 2\int_0^1 ue^u\,du$.

Third Figure Let $u = \sin x$, so $du = \cos x\,dx$. When $x=0$, $u=0$; when $x=\frac{\pi}{2}$, $u=1$. Thus,

$$A_3 = \int_0^{\pi/2} e^{\sin x}\sin 2x\,dx = \int_0^{\pi/2} e^{\sin x}(2\sin x\,\cos x)\,dx = \int_0^1 e^u(2u\,du) = 2\int_0^1 ue^u\,du.$$

Since $A_1 = A_2 = A_3$, all three areas are equal.

62. Let $u = \dfrac{\pi t}{12}$. Then $du = \dfrac{\pi}{12}\,dt$ and

$$\int_0^{24} R(t)\,dt = \int_0^{24}\left[85 - 0.18\cos\left(\frac{\pi t}{12}\right)\right]dt = \int_0^{2\pi}(85 - 0.18\cos u)\left(\frac{12}{\pi}\,du\right) = \frac{12}{\pi}\left[85u - 0.18\sin u\right]_0^{2\pi}$$
$$= \frac{12}{\pi}\left[(85\cdot 2\pi - 0) - (0-0)\right] = 2040 \text{ kcal}$$

63. The rate is measured in liters per minute. Integrating from $t=0$ minutes to $t=60$ minutes will give us the total amount of oil that leaks out (in liters) during the first hour.

$$\int_0^{60} r(t)\,dt = \int_0^{60} 100e^{-0.01t}\,dt \qquad [u = -0.01t,\, du = -0.01dt]$$
$$= 100\int_0^{-0.6} e^u(-100\,du) = -10{,}000\left[e^u\right]_0^{-0.6} = -10{,}000(e^{-0.6}-1) \approx 4511.9 \approx 4512 \text{ liters}$$

64. Let $r(t) = ae^{bt}$ with $a = 450.268$ and $b = 1.12567$, and $n(t) =$ population after t hours. Since $r(t) = n'(t)$, $\int_0^3 r(t)\,dt = n(3) - n(0)$ is the total change in the population after three hours. Since we start with 400 bacteria, the population will be

$$n(3) = 400 + \int_0^3 r(t)\,dt = 400 + \int_0^3 ae^{bt}\,dt = 400 + \frac{a}{b}\left[e^{bt}\right]_0^3 = 400 + \frac{a}{b}\left(e^{3b}-1\right)$$
$$\approx 400 + 11{,}313 = 11{,}713 \text{ bacteria}$$

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65. The volume of inhaled air in the lungs at time t is

$$V(t) = \int_0^t f(u)\,du = \int_0^t \frac{1}{2}\sin\left(\frac{2\pi}{5}u\right)du = \int_0^{2\pi t/5} \frac{1}{2}\sin v\left(\frac{5}{2\pi}\,dv\right) \qquad \left[\text{substitute } v = \tfrac{2\pi}{5}u,\ dv = \tfrac{2\pi}{5}\,du\right]$$

$$= \frac{5}{4\pi}\big[-\cos v\big]_0^{2\pi t/5} = \frac{5}{4\pi}\left[-\cos\left(\frac{2\pi}{5}t\right)+1\right] = \frac{5}{4\pi}\left[1-\cos\left(\frac{2\pi}{5}t\right)\right] \text{ liters}$$

66. Number of calculators $= x(4) - x(2) = \int_2^4 5000\left[1 - 100(t+10)^{-2}\right]dt$

$$= 5000\left[t + 100(t+10)^{-1}\right]_2^4 = 5000\left[\left(4 + \tfrac{100}{14}\right) - \left(2 + \tfrac{100}{12}\right)\right] \approx 4048$$

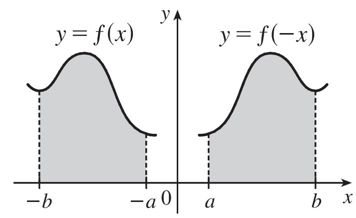
67. Let $u = 2x$. Then $du = 2\,dx$, so $\int_0^2 f(2x)\,dx = \int_0^4 f(u)\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int_0^4 f(u)\,du = \frac{1}{2}(10) = 5$.

68. Let $u = x^2$. Then $du = 2x\,dx$, so $\int_0^3 x f(x^2)\,dx = \int_0^9 f(u)\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int_0^9 f(u)\,du = \frac{1}{2}(4) = 2$.

69. Let $u = -x$. Then $du = -dx$, so

$$\int_a^b f(-x)\,dx = \int_{-a}^{-b} f(u)(-du) = \int_{-b}^{-a} f(u)\,du = \int_{-b}^{-a} f(x)\,dx$$

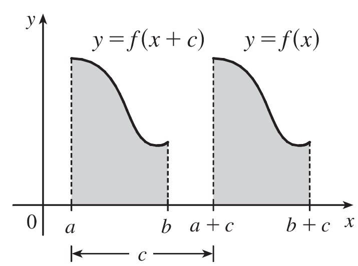
From the diagram, we see that the equality follows from the fact that we are reflecting the graph of f, and the limits of integration, about the y-axis.



70. Let $u = x + c$. Then $du = dx$, so

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(u)\,du = \int_{a+c}^{b+c} f(x)\,dx$$

From the diagram, we see that the equality follows from the fact that we are translating the graph of f, and the limits of integration, by a distance c.



71. Let $u = 1 - x$. Then $x = 1 - u$ and $dx = -du$, so

$$\int_0^1 x^a(1-x)^b\,dx = \int_1^0 (1-u)^a\,u^b(-du) = \int_0^1 u^b(1-u)^a\,du = \int_0^1 x^b(1-x)^a\,dx.$$

72. (a) $\int_0^{\pi/2} f(\cos x)\,dx = \int_0^{\pi/2} f\left[\sin\left(\frac{\pi}{2} - x\right)\right]dx \qquad \left[u = \frac{\pi}{2} - x,\ du = -dx\right]$

$$= \int_{\pi/2}^0 f(\sin u)(-du) = \int_0^{\pi/2} f(\sin u)\,du = \int_0^{\pi/2} f(\sin x)\,dx$$

Continuity of f is needed in order to apply the substitution rule for definite integrals.

(b) In part (a), take $f(x) = x^2$, so $\int_0^{\pi/2}\cos^2 x\,dx = \int_0^{\pi/2}\sin^2 x\,dx$. Now

$$\int_0^{\pi/2}\cos^2 x\,dx + \int_0^{\pi/2}\sin^2 x\,dx = \int_0^{\pi/2}(\cos^2 x + \sin^2 x)\,dx = \int_0^{\pi/2} 1\,dx = \big[x\big]_0^{\pi/2} = \frac{\pi}{2},$$

so $2\int_0^{\pi/2}\cos^2 x\,dx = \frac{\pi}{2} \;\Rightarrow\; \int_0^{\pi/2}\cos^2 x\,dx = \frac{\pi}{4} \quad \left[= \int_0^{\pi/2}\sin^2 x\,dx\right]$.

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5.6 Integration by Parts

1. Let $u = \ln x$, $dv = x^2\,dx \quad\Rightarrow\quad du = \frac{1}{x}\,dx$, $v = \frac{1}{3}x^3$. Then by Equation 2,

$$\int x^2 \ln x\,dx = (\ln x)\left(\tfrac{1}{3}x^3\right) - \int \left(\tfrac{1}{3}x^3\right)\left(\tfrac{1}{x}\right)\,dx = \tfrac{1}{3}x^3\,\ln x - \tfrac{1}{3}\int x^2\,dx = \tfrac{1}{3}x^3\,\ln x - \tfrac{1}{3}\left(\tfrac{1}{3}x^3\right) + C$$

$$= \tfrac{1}{3}x^3\,\ln x - \tfrac{1}{9}x^3 + C \quad \left[\text{or } \tfrac{1}{3}x^3\left(\ln x - \tfrac{1}{3}\right) + C\right]$$

2. Let $u = \theta$, $dv = \cos\theta\,d\theta \quad\Rightarrow\quad du = d\theta$, $v = \sin\theta$. Then by Equation 2,

$$\int \theta\cos\theta\,d\theta = \theta\sin\theta - \int \sin\theta\,d\theta = \theta\sin\theta + \cos\theta + C.$$

Note: A mnemonic device which is helpful for selecting u when using integration by parts is the LIATE principle of precedence for u:

Logarithmic
Inverse trigonometric
Algebraic
Trigonometric
Exponential

If the integrand has several factors, then we try to choose among them a u which appears as high as possible on the list. For example, in $\int xe^{2x}\,dx$ the integrand is xe^{2x}, which is the product of an algebraic function (x) and an exponential function (e^{2x}). Since Algebraic appears before Exponential, we choose $u = x$. Sometimes the integration turns out to be similar regardless of the selection of u and dv, but it is advisable to refer to LIATE when in doubt.

3. Let $u = x$, $dv = \cos 5x\,dx \quad\Rightarrow\quad du = dx$, $v = \frac{1}{5}\sin 5x$. Then by Equation 2,

$$\int x\cos 5x\,dx = \tfrac{1}{5}x\sin 5x - \int \tfrac{1}{5}\sin 5x\,dx = \tfrac{1}{5}x\sin 5x + \tfrac{1}{25}\cos 5x + C.$$

4. Let $u = x$, $dv = e^{-x}\,dx \quad\Rightarrow\quad du = dx$, $v = -e^{-x}$.

Then $\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C = -(x+1)e^{-x} + C$.

5. Let $u = r$, $dv = e^{r/2}\,dr \quad\Rightarrow\quad du = dr$, $v = 2e^{r/2}$.

Then $\int re^{r/2}\,dr = 2re^{r/2} - \int 2e^{r/2}\,dr = 2re^{r/2} - 4e^{r/2} + C = 2(r-2)e^{r/2} + C$.

6. Let $u = t$, $dv = \sin 2t\,dt \quad\Rightarrow\quad du = dt$, $v = -\frac{1}{2}\cos 2t$. Then

$$\int t\sin 2t\,dt = -\tfrac{1}{2}t\cos 2t + \tfrac{1}{2}\int \cos 2t\,dt = -\tfrac{1}{2}t\cos 2t + \tfrac{1}{4}\sin 2t + C.$$

7. Let $u = x^2$, $dv = \sin\pi x\,dx \quad\Rightarrow\quad du = 2x\,dx$ and $v = -\frac{1}{\pi}\cos\pi x$. Then

$I = \int x^2\,\sin\pi x\,dx = -\frac{1}{\pi}x^2\cos\pi x + \frac{2}{\pi}\int x\cos\pi x\,dx$ $(\star)$. Next let $U = x$, $dV = \cos\pi x\,dx \quad\Rightarrow\quad dU = dx$,

$V = \frac{1}{\pi}\sin\pi x$, so $\int x\cos\pi x\,dx = \frac{1}{\pi}x\sin\pi x - \frac{1}{\pi}\int \sin\pi x\,dx = \frac{1}{\pi}x\sin\pi x + \frac{1}{\pi^2}\cos\pi x + C_1$.

Substituting for $\int x\cos\pi x\,dx$ in $(\star)$, we get

$I = -\frac{1}{\pi}x^2\cos\pi x + \frac{2}{\pi}\left(\frac{1}{\pi}x\sin\pi x + \frac{1}{\pi^2}\cos\pi x + C_1\right) = -\frac{1}{\pi}x^2\cos\pi x + \frac{2}{\pi^2}x\sin\pi x + \frac{2}{\pi^3}\cos\pi x + C$, where $C = \frac{2}{\pi}C_1$.

8. Let $u = x^2$, $dv = \cos mx\,dx \quad\Rightarrow\quad du = 2x\,dx$, $v = \dfrac{1}{m}\sin mx$. Then

$I = \displaystyle\int x^2\,\cos mx\,dx = \frac{1}{m}x^2\,\sin mx - \frac{2}{m}\int x\sin mx\,dx$ $(\star)$. Next let $U = x$, $dV = \sin mx\,dx \quad\Rightarrow\quad dU = dx$,

$V = -\dfrac{1}{m}\cos mx$, so $\displaystyle\int x\sin mx\,dx = -\frac{1}{m}x\cos mx + \frac{1}{m}\int \cos mx\,dx = -\frac{1}{m}x\cos mx + \frac{1}{m^2}\sin mx + C_1$.

Substituting for $\int x\sin mx\,dx$ in $(\star)$, we get

$$I = \frac{1}{m}x^2\,\sin mx - \frac{2}{m}\left(-\frac{1}{m}x\cos mx + \frac{1}{m^2}\sin mx + C_1\right) = \frac{1}{m}x^2\,\sin mx + \frac{2}{m^2}x\cos mx - \frac{2}{m^3}\sin mx + C,$$

where $C = -\dfrac{2}{m}C_1$.

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9. Let $u = \ln \sqrt[3]{x}$, $dv = dx \quad \Rightarrow \quad du = \dfrac{1}{\sqrt[3]{x}}\left(\dfrac{1}{3}x^{-2/3}\right)dx = \dfrac{1}{3x}\,dx$, $v = x$. Then

$$\int \ln \sqrt[3]{x}\,dx = x\ln\sqrt[3]{x} - \int x \cdot \frac{1}{3x}dx = x\ln\sqrt[3]{x} - \frac{1}{3}x + C.$$

Second solution: Rewrite $\int \ln \sqrt[3]{x}\,dx = \frac{1}{3}\int \ln x\,dx$, and apply Example 2.

Third solution: Substitute $y = \sqrt[3]{x}$, to obtain $\int \ln \sqrt[3]{x}\,dx = 3\int y^2 \ln y\,dy$, and apply Exercise 1.

10. Let $u = \ln p$, $dv = p^5 dp \quad \Rightarrow \quad du = \dfrac{1}{p}\,dp$, $v = \frac{1}{6}p^6$. Then $\int p^5 \ln p\,dp = \frac{1}{6}p^6 \ln p - \frac{1}{6}\int p^5 dp = \frac{1}{6}p^6 \ln p - \frac{1}{36}p^6 + C$.

11. Let $u = \arctan 4t$, $dv = dt \quad \Rightarrow \quad du = \dfrac{4}{1+(4t)^2}\,dt = \dfrac{4}{1+16t^2}\,dt$, $v = t$. Then

$$\int \arctan 4t\,dt = t\arctan 4t - \int \frac{4t}{1+16t^2}\,dt = t\arctan 4t - \frac{1}{8}\int \frac{32t}{1+16t^2}\,dt = t\arctan 4t - \tfrac{1}{8}\ln(1+16t^2) + C.$$

12. Let $u = \sin^{-1} x$, $dv = dx \quad \Rightarrow \quad du = \dfrac{dx}{\sqrt{1-x^2}}$, $v = x$. Then $\int \sin^{-1} x\,dx = x\sin^{-1} x - \displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$. Setting

$t = 1 - x^2$, we get $dt = -2x\,dx$, so $-\displaystyle\int \frac{x\,dx}{\sqrt{1-x^2}} = -\int t^{-1/2}\left(-\tfrac{1}{2}\,dt\right) = \tfrac{1}{2}(2t^{1/2}) + C = t^{1/2} + C = \sqrt{1-x^2} + C$.

Hence, $\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1-x^2} + C$.

13. First let $u = \sin 3\theta$, $dv = e^{2\theta}\,d\theta \quad \Rightarrow \quad du = 3\cos 3\theta\,d\theta$, $v = \frac{1}{2}e^{2\theta}$. Then

$I = \int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{2}\int e^{2\theta}\cos 3\theta\,d\theta$. Next let $U = \cos 3\theta$, $dV = e^{2\theta}\,d\theta \quad \Rightarrow \quad dU = -3\sin 3\theta\,d\theta$,

$V = \frac{1}{2}e^{2\theta}$ to get $\int e^{2\theta}\cos 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\cos 3\theta + \frac{3}{2}\int e^{2\theta}\sin 3\theta\,d\theta$. Substituting in the previous formula gives

$I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}\int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}I \quad \Rightarrow$

$\frac{13}{4}I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta + C_1$. Hence, $I = \frac{1}{13}e^{2\theta}(2\sin 3\theta - 3\cos 3\theta) + C$, where $C = \frac{4}{13}C_1$.

14. First let $u = e^{-\theta}$, $dv = \cos 2\theta\,d\theta \quad \Rightarrow \quad du = -e^{-\theta}\,d\theta$, $v = \frac{1}{2}\sin 2\theta$. Then

$I = \int e^{-\theta}\cos 2\theta\,d\theta = \frac{1}{2}e^{-\theta}\sin 2\theta - \int \frac{1}{2}\sin 2\theta\left(-e^{-\theta}\,d\theta\right) = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\int e^{-\theta}\sin 2\theta\,d\theta$.

Next let $U = e^{-\theta}$, $dV = \sin 2\theta\,d\theta \quad \Rightarrow \quad dU = -e^{-\theta}\,d\theta$, $V = -\frac{1}{2}\cos 2\theta$, so

$\int e^{-\theta}\sin 2\theta\,d\theta = -\frac{1}{2}e^{-\theta}\cos 2\theta - \int\left(-\frac{1}{2}\right)\cos 2\theta\left(-e^{-\theta}\,d\theta\right) = -\frac{1}{2}e^{-\theta}\cos 2\theta - \frac{1}{2}\int e^{-\theta}\cos 2\theta\,d\theta$.

So $I = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\left[\left(-\frac{1}{2}e^{-\theta}\cos 2\theta\right) - \frac{1}{2}I\right] = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta - \frac{1}{4}I \quad \Rightarrow$

$\frac{5}{4}I = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1 \quad \Rightarrow \quad I = \frac{4}{5}\left(\frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1\right) = \frac{2}{5}e^{-\theta}\sin 2\theta - \frac{1}{5}e^{-\theta}\cos 2\theta + C$.

15. Let $u = t$, $dv = \sin 3t\,dt \quad \Rightarrow \quad du = dt$, $v = -\frac{1}{3}\cos 3t$. Then

$\int_0^\pi t\sin 3t\,dt = \left[-\frac{1}{3}t\cos 3t\right]_0^\pi + \frac{1}{3}\int_0^\pi \cos 3t\,dt = \left(\frac{1}{3}\pi - 0\right) + \frac{1}{9}\left[\sin 3t\right]_0^\pi = \frac{\pi}{3}$.

16. First let $u = x^2 + 1$, $dv = e^{-x}\,dx \quad \Rightarrow \quad du = 2x\,dx$, $v = -e^{-x}$. By (6),

$\int_0^1 (x^2+1)e^{-x}\,dx = \left[-(x^2+1)e^{-x}\right]_0^1 + \int_0^1 2xe^{-x}\,dx = -2e^{-1} + 1 + 2\int_0^1 xe^{-x}\,dx$.

Next let $U = x$, $dV = e^{-x}\,dx \quad \Rightarrow \quad dU = dx$, $V = -e^{-x}$. By (6) again,

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$\int_0^1 xe^{-x}\,dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx = -e^{-1} + \left[-e^{-x}\right]_0^1 = -e^{-1} - e^{-1} + 1 = -2e^{-1} + 1$. So

$\int_0^1 (x^2+1)e^{-x}\,dx = -2e^{-1} + 1 + 2(-2e^{-1}+1) = -2e^{-1} + 1 - 4e^{-1} + 2 = -6e^{-1} + 3$.

17. Let $u = \ln x$, $dv = x^{-2}\,dx \quad\Rightarrow\quad du = \dfrac{1}{x}\,dx$, $v = -x^{-1}$. By (6),

$$\int_1^2 \frac{\ln x}{x^2}\,dx = \left[-\frac{\ln x}{x}\right]_1^2 + \int_1^2 x^{-2}\,dx = -\tfrac{1}{2}\ln 2 + \ln 1 + \left[-\frac{1}{x}\right]_1^2 = -\tfrac{1}{2}\ln 2 + 0 - \tfrac{1}{2} + 1 = \tfrac{1}{2} - \tfrac{1}{2}\ln 2.$$

18. Let $u = \ln y$, $dv = \dfrac{1}{\sqrt{y}}\,dy = y^{-1/2}\,dy \quad\Rightarrow\quad du = \dfrac{1}{y}\,dy$, $v = 2y^{1/2}$. Then

$$\int_4^9 \frac{\ln y}{\sqrt{y}}\,dy = \left[2\sqrt{y}\,\ln y\right]_4^9 - \int_4^9 2y^{-1/2}\,dy = (6\ln 9 - 4\ln 4) - \left[4\sqrt{y}\right]_4^9 = 6\ln 9 - 4\ln 4 - (12-8)$$
$$= 6\ln 9 - 4\ln 4 - 4$$

19. Let $u = y$, $dv = \dfrac{dy}{e^{2y}} = e^{-2y}dy \quad\Rightarrow\quad du = dy$, $v = -\frac{1}{2}e^{-2y}$. Then

$$\int_0^1 \frac{y}{e^{2y}}\,dy = \left[-\tfrac{1}{2}ye^{-2y}\right]_0^1 + \tfrac{1}{2}\int_0^1 e^{-2y}dy = \left(-\tfrac{1}{2}e^{-2} + 0\right) - \tfrac{1}{4}\left[e^{-2y}\right]_0^1 = -\tfrac{1}{2}e^{-2} - \tfrac{1}{4}e^{-2} + \tfrac{1}{4} = \tfrac{1}{4} - \tfrac{3}{4}e^{-2}.$$

20. Let $u = \arctan(1/x)$, $dv = dx \quad\Rightarrow\quad du = \dfrac{1}{1+(1/x)^2}\cdot\dfrac{-1}{x^2}\,dx = \dfrac{-dx}{x^2+1}$, $v = x$. Then

$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right)dx = \left[x\arctan\left(\frac{1}{x}\right)\right]_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x\,dx}{x^2+1} = \sqrt{3}\,\frac{\pi}{6} - 1\cdot\frac{\pi}{4} + \frac{1}{2}\left[\ln(x^2+1)\right]_1^{\sqrt{3}}$$
$$= \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}(\ln 4 - \ln 2) = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}\ln\frac{4}{2} = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}\ln 2$$

21. Let $u = \cos^{-1}x$, $dv = dx \quad\Rightarrow\quad du = -\dfrac{dx}{\sqrt{1-x^2}}$, $v = x$. Then

$$I = \int_0^{1/2} \cos^{-1}x\,dx = \left[x\cos^{-1}x\right]_0^{1/2} + \int_0^{1/2} \frac{x\,dx}{\sqrt{1-x^2}} = \tfrac{1}{2}\cdot\tfrac{\pi}{3} + \int_1^{3/4} t^{-1/2}\left[-\tfrac{1}{2}dt\right], \text{ where } t = 1 - x^2 \quad\Rightarrow$$

$dt = -2x\,dx$. Thus, $I = \frac{\pi}{6} + \frac{1}{2}\int_{3/4}^{1} t^{-1/2}\,dt = \frac{\pi}{6} + \left[\sqrt{t}\right]_{3/4}^{1} = \frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2} = \frac{1}{6}\left(\pi + 6 - 3\sqrt{3}\right)$.

22. Let $u = r^2$, $dv = \dfrac{r}{\sqrt{4+r^2}}\,dr \quad\Rightarrow\quad du = 2r\,dr$, $v = \sqrt{4+r^2}$. By (6),

$$\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr = \left[r^2\sqrt{4+r^2}\right]_0^1 - 2\int_0^1 r\sqrt{4+r^2}\,dr = \sqrt{5} - \tfrac{2}{3}\left[(4+r^2)^{3/2}\right]_0^1$$
$$= \sqrt{5} - \tfrac{2}{3}(5)^{3/2} + \tfrac{2}{3}(8) = \sqrt{5}\left(1 - \tfrac{10}{3}\right) + \tfrac{16}{3} = \tfrac{16}{3} - \tfrac{7}{3}\sqrt{5}$$

23. Let $u = (\ln x)^2$, $dv = dx \quad\Rightarrow\quad du = \dfrac{2}{x}\ln x\,dx$, $v = x$. By (6), $I = \int_1^2 (\ln x)^2\,dx = \left[x(\ln x)^2\right]_1^2 - 2\int_1^2 \ln x\,dx$.

To evaluate the last integral, let $U = \ln x$, $dV = dx \quad\Rightarrow\quad dU = \dfrac{1}{x}\,dx$, $V = x$. Thus,

$$I = \left[x(\ln x)^2\right]_1^2 - 2\left(\left[x\ln x\right]_1^2 - \int_1^2 dx\right) = \left[x(\ln x)^2 - 2x\ln x + 2x\right]_1^2$$
$$= \left(2(\ln 2)^2 - 4\ln 2 + 4\right) - (0 - 0 + 2) = 2(\ln 2)^2 - 4\ln 2 + 2$$

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24. Let $u = \sin(t-s)$, $dv = e^s\,ds \;\Rightarrow\; du = -\cos(t-s)\,ds$, $v = e^s$. Then
$I = \int_0^t e^s \sin(t-s)\,ds = \Big[e^s\sin(t-s)\Big]_0^t + \int_0^t e^s\cos(t-s)\,ds = e^t\sin 0 - e^0\sin t + I_1$. For I_1, let $U = \cos(t-s)$, $dV = e^s\,ds \;\Rightarrow\; dU = \sin(t-s)\,ds$, $V = e^s$. So $I_1 = \big[e^s\cos(t-s)\big]_0^t - \int_0^t e^s\sin(t-s)\,ds = e^t\cos 0 - e^0\cos t - I$.
Thus, $I = -\sin t + e^t - \cos t - I \;\Rightarrow\; 2I = e^t - \cos t - \sin t \;\Rightarrow\; I = \frac{1}{2}(e^t - \cos t - \sin t)$.

25. Let $y = \sqrt{x}$, so that $dy = \frac{1}{2}x^{-1/2}\,dx = \dfrac{1}{2\sqrt{x}}\,dx = \dfrac{1}{2y}\,dx$. Thus, $\int \cos\sqrt{x}\,dx = \int \cos y\,(2y\,dy) = 2\int y\cos y\,dy$. Now use parts with $u = y$, $dv = \cos y\,dy$, $du = dy$, $v = \sin y$ to get $\int y\cos y\,dy = y\sin y - \int \sin y\,dy = y\sin y + \cos y + C_1$, so $\int \cos\sqrt{x}\,dx = 2y\sin y + 2\cos y + C = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$.

26. Let $x = -t^2$, so that $dx = -2t\,dt$. Thus, $\int t^3 e^{-t^2}\,dt = \int (-t^2)e^{-t^2}\left(\frac{1}{2}\right)(-2t\,dt) = \frac{1}{2}\int xe^x\,dx$. Now use parts with $u = x$, $dv = e^x\,dx$, $du = dx$, $v = e^x$ to get
$\frac{1}{2}\int xe^x\,dx = \frac{1}{2}\left(xe^x - \int e^x\,dx\right) = \frac{1}{2}xe^x - \frac{1}{2}e^x + C = -\frac{1}{2}(1-x)e^x + C = -\frac{1}{2}(1+t^2)e^{-t^2} + C$.

27. Let $x = \theta^2$, so that $dx = 2\theta\,d\theta$. Thus, $\displaystyle\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3\cos\left(\theta^2\right)\,d\theta = \int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^2\cos\left(\theta^2\right)\cdot\tfrac{1}{2}(2\theta\,d\theta) = \tfrac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx$. Now use parts with $u = x$, $dv = \cos x\,dx$, $du = dx$, $v = \sin x$ to get

$$\begin{aligned}\tfrac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx &= \tfrac{1}{2}\left(\big[x\sin x\big]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}\sin x\,dx\right) = \tfrac{1}{2}\big[x\sin x + \cos x\big]_{\pi/2}^{\pi} \\ &= \tfrac{1}{2}(\pi\sin\pi + \cos\pi) - \tfrac{1}{2}\left(\tfrac{\pi}{2}\sin\tfrac{\pi}{2} + \cos\tfrac{\pi}{2}\right) = \tfrac{1}{2}(\pi\cdot 0 - 1) - \tfrac{1}{2}\left(\tfrac{\pi}{2}\cdot 1 + 0\right) = -\tfrac{1}{2} - \tfrac{\pi}{4}\end{aligned}$$

28. Let $x = \cos t$, so that $dx = -\sin t\,dt$. Thus,
$\int_0^\pi e^{\cos t}\sin 2t\,dt = \int_0^\pi e^{\cos t}(2\sin t\,\cos t)\,dt = \int_1^{-1} e^x\cdot 2x\,(-dx) = 2\int_{-1}^1 xe^x\,dx$. Now use parts with $u = x$, $dv = e^x\,dx$, $du = dx$, $v = e^x$ to get
$2\int_{-1}^1 xe^x\,dx = 2\left(\big[xe^x\big]_{-1}^1 - \int_{-1}^1 e^x\,dx\right) = 2\left(e^1 + e^{-1} - \big[e^x\big]_{-1}^1\right) = 2(e + e^{-1} - [e^1 - e^{-1}]) = 2(2e^{-1}) = 4/e$.

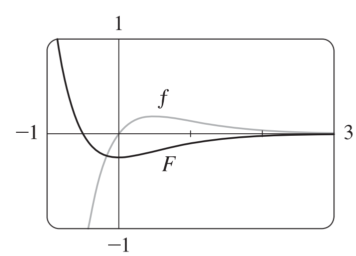
29. Let $y = 1 + x$, so that $dy = dx$. Thus, $\int x\ln(1+x)\,dx = \int (y-1)\ln y\,dy$. Now use parts with $u = \ln y$, $dv = (y-1)\,dy$, $du = \frac{1}{y}\,dy$, $v = \frac{1}{2}y^2 - y$ to get

$$\begin{aligned}\int (y-1)\ln y\,dy &= \left(\tfrac{1}{2}y^2 - y\right)\ln y - \int\left(\tfrac{1}{2}y - 1\right)dy = \tfrac{1}{2}y(y-2)\ln y - \tfrac{1}{4}y^2 + y + C \\ &= \tfrac{1}{2}(1+x)(x-1)\ln(1+x) - \tfrac{1}{4}(1+x)^2 + 1 + x + C,\end{aligned}$$

which can be written as $\frac{1}{2}(x^2-1)\ln(1+x) - \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{4} + C$.

30. Let $y = \ln x$, so that $dy = \frac{1}{x}\,dx \;\Rightarrow\; dx = x\,dy = e^y\,dy$. Thus,
$\int \sin(\ln x)\,dx = \int \sin y\,e^y\,dy = \frac{1}{2}e^y(\sin y - \cos y) + C$ [by Example 4] $= \frac{1}{2}x[\sin(\ln x) - \cos(\ln x)] + C$.

31. Let $u = x$, $dv = e^{-2x}\,dx \;\Rightarrow\; du = dx$, $v = -\frac{1}{2}e^{-2x}$. Then
$\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} + \int \frac{1}{2}e^{-2x}\,dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$. We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive. Also, F increases where f is positive and F decreases where f is negative.



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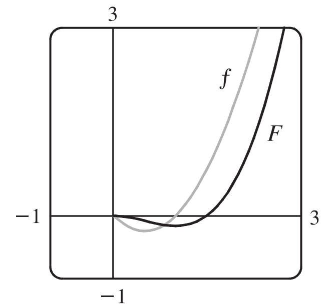
32. Let $u = \ln x$, $dv = x^{3/2}\,dx \;\Rightarrow\; du = \frac{1}{x}\,dx$, $v = \frac{2}{5}x^{5/2}$. Then

$$\int x^{3/2}\ln x\,dx = \tfrac{2}{5}x^{5/2}\ln x - \tfrac{2}{5}\int x^{3/2}\,dx = \tfrac{2}{5}x^{5/2}\ln x - \left(\tfrac{2}{5}\right)^2 x^{5/2} + C$$
$$= \tfrac{2}{5}x^{5/2}\ln x - \tfrac{4}{25}x^{5/2} + C$$

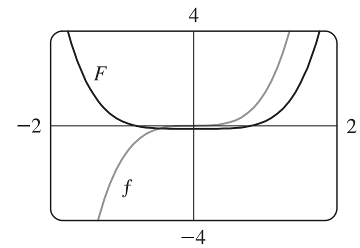
We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.



33. Let $u = \frac{1}{2}x^2$, $dv = 2x\sqrt{1+x^2}\,dx \;\Rightarrow\; du = x\,dx$, $v = \frac{2}{3}(1+x^2)^{3/2}$.

Then

$$\int x^3\sqrt{1+x^2}\,dx = \tfrac{1}{2}x^2\left[\tfrac{2}{3}(1+x^2)^{3/2}\right] - \tfrac{2}{3}\int x(1+x^2)^{3/2}dx$$
$$= \tfrac{1}{3}x^2(1+x^2)^{3/2} - \tfrac{2}{3}\cdot\tfrac{2}{5}\cdot\tfrac{1}{2}(1+x^2)^{5/2} + C$$
$$= \tfrac{1}{3}x^2(1+x^2)^{3/2} - \tfrac{2}{15}(1+x^2)^{5/2} + C$$



We see from the graph that this is reasonable, since F increases where f is positive and F decreases where f is negative. Note also that f is an odd function and F is an even function.

Another method: Use substitution with $u = 1 + x^2$ to get $\frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + C$.

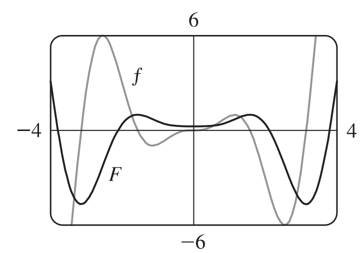
34. First let $u = x^2$, $dv = \sin 2x\,dx \;\Rightarrow\; du = 2x\,dx$, $v = -\frac{1}{2}\cos 2x$.

Then $I = \int x^2\sin 2x\,dx = -\frac{1}{2}x^2\cos 2x + \int x\cos 2x\,dx$.

Next let $U = x$, $dV = \cos 2x\,dx \;\Rightarrow\; dU = dx$, $V = \frac{1}{2}\sin 2x$, so

$\int x\cos 2x\,dx = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$.

Thus, $I = -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$.



We see from the graph that this is reasonable, since F increases where f is positive and F decreases where f is negative. Note also that f is an odd function and F is an even function.

35. (a) Take $n = 2$ in Example 6 to get $\displaystyle\int \sin^2 x\,dx = -\frac{1}{2}\cos x\sin x + \frac{1}{2}\int 1\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$.

(b) $\int \sin^4 x\,dx = -\frac{1}{4}\cos x\sin^3 x + \frac{3}{4}\int \sin^2 x\,dx = -\frac{1}{4}\cos x\sin^3 x + \frac{3}{8}x - \frac{3}{16}\sin 2x + C$.

36. (a) Let $u = \cos^{n-1} x$, $dv = \cos x\,dx \;\Rightarrow\; du = -(n-1)\cos^{n-2} x\,\sin x\,dx$, $v = \sin x$ in (2):

$$\int \cos^n x\,dx = \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,\sin^2 x\,dx$$
$$= \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,(1-\cos^2 x)dx$$
$$= \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

Rearranging terms gives $n\int \cos^n x\,dx = \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,dx$ or

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\,\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

(b) Take $n = 2$ in part (a) to get $\int \cos^2 x\,dx = \frac{1}{2}\cos x\,\sin x + \frac{1}{2}\int 1\,dx = \dfrac{x}{2} + \dfrac{\sin 2x}{4} + C$.

(c) $\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x\,\sin x + \frac{3}{4}\int \cos^2 x\,dx = \frac{1}{4}\cos^3 x\,\sin x + \frac{3}{8}x + \frac{3}{16}\sin 2x + C$

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37. (a) From Example 6, $\int \sin^n x\,dx = -\frac{1}{n}\cos x\,\sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$. Using (6),

$$\int_0^{\pi/2} \sin^n x\,dx = \left[-\frac{\cos x\,\sin^{n-1} x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$
$$= (0-0) + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

(b) Using $n = 3$ in part (a), we have $\int_0^{\pi/2} \sin^3 x\,dx = \frac{2}{3}\int_0^{\pi/2} \sin x\,dx = \left[-\frac{2}{3}\cos x\right]_0^{\pi/2} = \frac{2}{3}$.

Using $n = 5$ in part (a), we have $\int_0^{\pi/2} \sin^5 x\,dx = \frac{4}{5}\int_0^{\pi/2} \sin^3 x\,dx = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15}$.

(c) The formula holds for $n = 1$ (that is, $2n+1 = 3$) by (b). Assume it holds for some $k \geq 1$. Then

$\int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)}$. By Example 6,

$$\int_0^{\pi/2} \sin^{2k+3} x\,dx = \frac{2k+2}{2k+3}\int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2k+2}{2k+3}\cdot\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)}$$
$$= \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)[2\,(k+1)]}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)[2\,(k+1)+1]},$$

so the formula holds for $n = k+1$. By induction, the formula holds for all $n \geq 1$.

38. Using Exercise 37(a), we see that the formula holds for $n = 1$, because $\int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{2}\int_0^{\pi/2} 1\,dx = \frac{1}{2}\left[x\right]_0^{\pi/2} = \frac{1}{2}\cdot\frac{\pi}{2}$.

Now assume it holds for some $k \geq 1$. Then $\int_0^{\pi/2} \sin^{2k} x\,dx = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}\frac{\pi}{2}$. By Exercise 37(a),

$$\int_0^{\pi/2} \sin^{2(k+1)} x\,dx = \frac{2k+1}{2k+2}\int_0^{\pi/2} \sin^{2k} x\,dx = \frac{2k+1}{2k+2}\cdot\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}\frac{\pi}{2}$$
$$= \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)(2k+1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)(2k+2)}\cdot\frac{\pi}{2},$$

so the formula holds for $n = k+1$. By induction, the formula holds for all $n \geq 1$.

39. Let $u = (\ln x)^n$, $dv = dx \quad\Rightarrow\quad du = n(\ln x)^{n-1}(dx/x)$, $v = x$. By Equation 2,

$\int(\ln x)^n\,dx = x(\ln x)^n - \int nx(\ln x)^{n-1}(dx/x) = x(\ln x)^n - n\int(\ln x)^{n-1}\,dx$.

40. Let $u = x^n$, $dv = e^x\,dx \quad\Rightarrow\quad du = nx^{n-1}\,dx$, $v = e^x$. By Equation 2, $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$.

41. By repeated applications of the reduction formula in Exercise 39,

$$\int(\ln x)^3\,dx = x\,(\ln x)^3 - 3\int(\ln x)^2\,dx = x(\ln x)^3 - 3\left[x(\ln x)^2 - 2\int(\ln x)^1\,dx\right]$$
$$= x\,(\ln x)^3 - 3x(\ln x)^2 + 6\left[x(\ln x)^1 - 1\int(\ln x)^0\,dx\right]$$
$$= x\,(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6\int 1\,dx = x\,(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C$$

42. By repeated applications of the reduction formula in Exercise 40,

$$\int x^4e^x\,dx = x^4e^x - 4\int x^3e^x\,dx = x^4e^x - 4\left(x^3e^x - 3\int x^2e^x\,dx\right)$$
$$= x^4e^x - 4x^3e^x + 12\left(x^2e^x - 2\int x^1e^x\,dx\right) = x^4e^x - 4x^3e^x + 12x^2e^x - 24\left(x^1e^x - \int x^0e^x\,dx\right)$$
$$= x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + C \quad \left[\text{or } e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) + C\right]$$

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43. Since $v(t) > 0$ for all t, the desired distance is $s(t) = \int_0^t v(w)\,dw = \int_0^t w^2 e^{-w}\,dw$.

First let $u = w^2$, $dv = e^{-w}\,dw \Rightarrow du = 2w\,dw$, $v = -e^{-w}$. Then $s(t) = \left[-w^2 e^{-w}\right]_0^t + 2\int_0^t we^{-w}\,dw$.

Next let $U = w$, $dV = e^{-w}\,dw \Rightarrow dU = dw$, $V = -e^{-w}$. Then

$$s(t) = -t^2e^{-t} + 2\left(\left[-we^{-w}\right]_0^t + \int_0^t e^{-w}\,dw\right) = -t^2e^{-t} + 2\left(-te^{-t} + 0 + \left[-e^{-w}\right]_0^t\right)$$
$$= -t^2e^{-t} + 2(-te^{-t} - e^{-t} + 1) = -t^2e^{-t} - 2te^{-t} - 2e^{-t} + 2 = 2 - e^{-t}(t^2 + 2t + 2) \text{ meters}$$

44. The rocket will have height $H = \int_0^{60} v(t)\,dt$ after 60 seconds.

$$H = \int_0^{60}\left[-gt - v_e \ln\left(\frac{m - rt}{m}\right)\right] dt = -g\left[\tfrac{1}{2}t^2\right]_0^{60} - v_e\left[\int_0^{60}\ln(m - rt)\,dt - \int_0^{60}\ln m\,dt\right]$$
$$= -g(1800) + v_e(\ln m)(60) - v_e\int_0^{60}\ln(m - rt)\,dt$$

Let $u = \ln(m - rt)$, $dv = dt \Rightarrow du = \dfrac{1}{m - rt}(-r)\,dt$, $v = t$. Then

$$\int_0^{60}\ln(m - rt)\,dt = \Big[t\ln(m - rt)\Big]_0^{60} + \int_0^{60}\frac{rt}{m - rt}\,dt = 60\ln(m - 60r) + \int_0^{60}\left(-1 + \frac{m}{m - rt}\right)dt$$
$$= 60\ln(m - 60r) + \left[-t - \frac{m}{r}\ln(m - rt)\right]_0^{60} = 60\ln(m - 60r) - 60 - \frac{m}{r}\ln(m - 60r) + \frac{m}{r}\ln m$$

So $H = -1800g + 60v_e\ln m - 60v_e\ln(m - 60r) + 60v_e + \dfrac{m}{r}v_e\ln(m - 60r) - \dfrac{m}{r}v_e\ln m$. Substituting $g = 9.8$, $m = 30{,}000$, $r = 160$, and $v_e = 3000$ gives us $H \approx 14{,}844$ m.

45. For $I = \int_1^4 xf''(x)\,dx$, let $u = x$, $dv = f''(x)\,dx \Rightarrow du = dx$, $v = f'(x)$. Then

$I = \left[xf'(x)\right]_1^4 - \int_1^4 f'(x)\,dx = 4f'(4) - 1\cdot f'(1) - [f(4) - f(1)] = 4\cdot 3 - 1\cdot 5 - (7 - 2) = 12 - 5 - 5 = 2.$

We used the fact that f'' is continuous to guarantee that I exists.

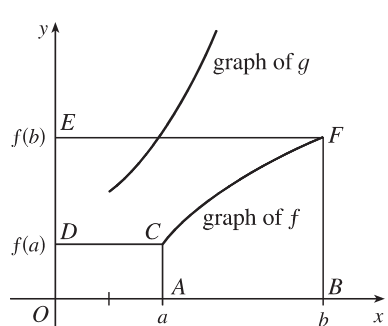
46. (a) Take $g(x) = x$ and $g'(x) = 1$ in Equation 1.

(b) By part (a), $\int_a^b f(x)\,dx = bf(b) - a\,f(a) - \int_a^b x\,f'(x)\,dx$. Now let $y = f(x)$, so that $x = g(y)$ and $dy = f'(x)\,dx$.

Then $\int_a^b x\,f'(x)\,dx = \int_{f(a)}^{f(b)} g(y)\,dy$. The result follows.

(c) Part (b) says that the area of region $ABFC$ is

$$= \quad bf(b) \quad - \quad af(a) \quad - \quad \int_{f(a)}^{f(b)} g(y)\,dy$$
$$= (\text{area of rectangle } OBFE) - (\text{area of rectangle } OACD) - (\text{area of region } DCFE)$$



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(d) We have $f(x) = \ln x$, so $f^{-1}(x) = e^x$, and since $g = f^{-1}$, we have $g(y) = e^y$. By part (b),

$$\int_1^e \ln x\,dx = e\ln e - 1\ln 1 - \int_{\ln 1}^{\ln e} e^y\,dy = e - \int_0^1 e^y\,dy = e - \left[e^y\right]_0^1 = e - (e-1) = 1.$$

47. Suppose $f(0) = g(0) = 0$ and let $u = f(x)$, $dv = g''(x)\,dx \quad\Rightarrow\quad du = f'(x)\,dx$, $v = g'(x)$.

Then $\int_0^a f(x)\,g''(x)\,dx = \Big[f(x)\,g'(x)\Big]_0^a - \int_0^a f'(x)\,g'(x)\,dx = f(a)\,g'(a) - \int_0^a f'(x)\,g'(x)\,dx.$

Now let $U = f'(x)$, $dV = g'(x)\,dx \quad\Rightarrow\quad dU = f''(x)\,dx$ and $V = g(x)$, so

$\int_0^a f'(x)\,g'(x)\,dx = \Big[f'(x)\,g(x)\Big]_0^a - \int_0^a f''(x)\,g(x)\,dx = f'(a)\,g(a) - \int_0^a f''(x)\,g(x)\,dx.$

Combining the two results, we get $\int_0^a f(x)\,g''(x)\,dx = f(a)\,g'(a) - f'(a)\,g(a) + \int_0^a f''(x)\,g(x)\,dx.$

48. (a) We note that for $0 \le x \le \frac{\pi}{2}$, $0 \le \sin x \le 1$, so $\sin^{2n+2} x \le \sin^{2n+1} x \le \sin^{2n} x$. So by the second Comparison Property of the Integral, $I_{2n+2} \le I_{2n+1} \le I_{2n}$.

(b) Substituting directly into the result from Exercise 38, we get

$$\frac{I_{2n+2}}{I_{2n}} = \frac{\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot[2(n+1)-1]}{2\cdot 4\cdot 6\cdot\cdots\cdot[2(n+1)]}\dfrac{\pi}{2}}{\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}\dfrac{\pi}{2}} = \frac{2(n+1)-1}{2(n+1)} = \frac{2n+1}{2n+2}$$

(c) We divide the result from part (a) by I_{2n}. The inequalities are preserved since I_{2n} is positive: $\dfrac{I_{2n+2}}{I_{2n}} \le \dfrac{I_{2n+1}}{I_{2n}} \le \dfrac{I_{2n}}{I_{2n}}$.

Now from part (b), the left term is equal to $\dfrac{2n+1}{2n+2}$, so the expression becomes $\dfrac{2n+1}{2n+2} \le \dfrac{I_{2n+1}}{I_{2n}} \le 1$. Now

$\lim\limits_{n\to\infty} \dfrac{2n+1}{2n+2} = \lim\limits_{n\to\infty} 1 = 1$, so by the Squeeze Theorem, $\lim\limits_{n\to\infty} \dfrac{I_{2n+1}}{I_{2n}} = 1$.

(d) We substitute the results from Exercises 37 and 38 into the result from part (c):

$$1 = \lim_{n\to\infty}\frac{I_{2n+1}}{I_{2n}} = \lim_{n\to\infty}\frac{\dfrac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}}{\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}\dfrac{\pi}{2}} = \lim_{n\to\infty}\left[\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}\right]\left[\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}\left(\frac{2}{\pi}\right)\right]$$

$$= \lim_{n\to\infty}\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdot\frac{2}{\pi} \qquad \text{[rearrange terms]}$$

Multiplying both sides by $\frac{\pi}{2}$ gives us the *Wallis product*:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\cdots$$

(e) The area of the kth rectangle is k. At the $2n$th step, the area is increased from $2n-1$ to $2n$ by multiplying the width by $\dfrac{2n}{2n-1}$, and at the $(2n+1)$th step, the area is increased from $2n$ to $2n+1$ by multiplying the height by $\dfrac{2n+1}{2n}$. These two steps multiply the ratio of width to height by $\dfrac{2n}{2n-1}$ and $\dfrac{1}{(2n+1)/(2n)} = \dfrac{2n}{2n+1}$ respectively. So, by part (d), the limiting ratio is $\dfrac{2}{1}\cdot\dfrac{2}{3}\cdot\dfrac{4}{3}\cdot\dfrac{4}{5}\cdot\dfrac{6}{5}\cdot\dfrac{6}{7}\cdot\cdots = \dfrac{\pi}{2}$.

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