(c) $S_{10} = \frac{2}{5 \cdot 3}[f(0) + 4f(0.4) + 2f(0.8) + 4f(1.2) + 2f(1.6) + 4f(2) + 2f(2.4)$
$+ 4f(2.8) + 2f(3.2) + 4f(3.6) + f(4)]$

≈ 0.804896

15. $f(x) = \dfrac{\cos x}{x}, \Delta x = \dfrac{5-1}{8} = \dfrac{1}{2}$

(a) $T_8 = \frac{1}{2 \cdot 2}\left[f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + \cdots + 2f(4) + 2f\left(\frac{9}{2}\right) + f(5)\right] \approx -0.495333$

(b) $M_8 = \frac{1}{2}\left[f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right) + f\left(\frac{17}{4}\right) + f\left(\frac{19}{4}\right)\right] \approx -0.543321$

(c) $S_8 = \frac{1}{2 \cdot 3}\left[f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + 2f(4) + 4f\left(\frac{9}{2}\right) + f(5)\right] \approx -0.526123$

16. $f(x) = \ln\left(x^3 + 2\right), \Delta x = \dfrac{6-4}{10} = \dfrac{1}{5}$

(a) $T_{10} = \frac{1}{5 \cdot 2}\,[f(4) + 2f(4.2) + 2f(4.4) + \cdots + 2f(5.6) + 2f(5.8) + f(6)] \approx 9.649753$

(b) $M_{10} = \frac{1}{5}[f(4.1) + f(4.3) + \cdots + f(5.7) + f(5.9)] \approx 9.650912$

(c) $S_{10} = \frac{1}{5 \cdot 3}[f(4) + 4f(4.2) + 2f(4.4) + 4f(4.6) + 2f(4.8) + 4f(5) + 2f(5.2) + 4f(5.4)$
$+ 2f(5.6) + 4f(5.8) + f(6)]$

≈ 9.650526

17. $f(x) = \cos(x^2), \Delta x = \frac{1-0}{8} = \frac{1}{8}$

(a) $T_8 = \frac{1}{8 \cdot 2}\left\{f(0) + 2\left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right)\right] + f(1)\right\} \approx 0.902333$

$M_8 = \frac{1}{8}\left[f\left(\frac{1}{16}\right) + f\left(\frac{3}{16}\right) + f\left(\frac{5}{16}\right) + \cdots + f\left(\frac{15}{16}\right)\right] = 0.905620$

(b) $f(x) = \cos(x^2)$, $f'(x) = -2x\sin(x^2)$, $f''(x) = -2\sin(x^2) - 4x^2\cos(x^2)$. For $0 \le x \le 1$, sin and cos are positive, so $|f''(x)| = 2\sin(x^2) + 4x^2\cos(x^2) \le 2 \cdot 1 + 4 \cdot 1 \cdot 1 = 6$ since $\sin(x^2) \le 1$ and $\cos\left(x^2\right) \le 1$ for all x, and $x^2 \le 1$ for $0 \le x \le 1$. So for $n = 8$, we take $K = 6$, $a = 0$, and $b = 1$ in Theorem 3, to get $|E_T| \le 6 \cdot 1^3/(12 \cdot 8^2) = \frac{1}{128} = 0.0078125$ and $|E_M| \le \frac{1}{256} = 0.00390625$. [A better estimate is obtained by noting from a graph of f'' that $|f''(x)| \le 4$ for $0 \le x \le 1$.]

(c) Take $K = 6$ [as in part (b)] in Theorem 3. $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \le 0.0001 \quad\Leftrightarrow\quad \dfrac{6(1-0)^3}{12n^2} \le 10^{-4} \quad\Leftrightarrow$

$\dfrac{1}{2n^2} \le \dfrac{1}{10^4} \quad\Leftrightarrow\quad 2n^2 \ge 10^4 \quad\Leftrightarrow\quad n^2 \ge 5000 \quad\Leftrightarrow\quad n \ge 71$. Take $n = 71$ for T_n. For E_M, again take $K = 6$ in Theorem 3 to get $|E_M| \le 10^{-4} \quad\Leftrightarrow\quad 4n^2 \ge 10^4 \quad\Leftrightarrow\quad n^2 \ge 2500 \quad\Leftrightarrow\quad n \ge 50$. Take $n = 50$ for M_n.

18. $f(x) = e^{1/x}, \Delta x = \frac{2-1}{10} = \frac{1}{10}$

(a) $T_{10} = \frac{1}{10 \cdot 2}[f(1) + 2f(1.1) + 2f(1.2) + \cdots + 2f(1.9) + f(2)] \approx 2.021976$

$M_{10} = \frac{1}{10}\,[f(1.05) + f(1.15) + f(1.25) + \cdots + f(1.95)] \approx 2.019102$

(b) $f(x) = e^{1/x}$, $f'(x) = -\dfrac{1}{x^2}e^{1/x}$, $f''(x) = \dfrac{2x+1}{x^4}\,e^{1/x}$. Now f'' is decreasing on $[1, 2]$, so let $x = 1$ to take $K = 3e$.

$|E_T| \le \dfrac{3e(2-1)^3}{12(10)^2} = \dfrac{e}{400} \approx 0.006796. \quad |E_M| \le \dfrac{|E_T|}{2} = \dfrac{e}{800} \approx 0.003398.$

(c) Take $K = 3e$ [as in part (b)] in Theorem 3. $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \le 0.0001 \quad\Leftrightarrow\quad \dfrac{3e(2-1)^3}{12n^2} \le 10^{-4} \quad\Leftrightarrow$

$\dfrac{e}{4n^2} \le \dfrac{1}{10^4} \quad\Leftrightarrow\quad n^2 \ge \dfrac{10^4 e}{4} \quad\Leftrightarrow\quad n \ge 83$. Take $n = 83$ for T_n. For E_M, again take $K = 3e$ in Theorem 3 to get

$|E_M| \le 10^{-4} \quad\Leftrightarrow\quad n^2 \ge \dfrac{10^4 e}{8} \quad\Leftrightarrow\quad n \ge 59$. Take $n = 59$ for M_n.

19. $f(x) = \sin x$, $\Delta x = \frac{\pi - 0}{10} = \frac{\pi}{10}$

(a) $T_{10} = \frac{\pi}{10 \cdot 2}\left[f(0) + 2f\left(\frac{\pi}{10}\right) + 2f\left(\frac{2\pi}{10}\right) + \cdots + 2f\left(\frac{9\pi}{10}\right) + f(\pi)\right] \approx 1.983524$

$M_{10} = \frac{\pi}{10}\left[f\left(\frac{\pi}{20}\right) + f\left(\frac{3\pi}{20}\right) + f\left(\frac{5\pi}{20}\right) + \cdots + f\left(\frac{19\pi}{20}\right)\right] \approx 2.008248$

$S_{10} = \frac{\pi}{10 \cdot 3}\left[f(0) + 4f\left(\frac{\pi}{10}\right) + 2f\left(\frac{2\pi}{10}\right) + 4f\left(\frac{3\pi}{10}\right) + \cdots + 4f\left(\frac{9\pi}{10}\right) + f(\pi)\right] \approx 2.000110$

Since $I = \int_0^\pi \sin x\, dx = \left[-\cos x\right]_0^\pi = 1 - (-1) = 2$, $E_T = I - T_{10} \approx 0.016476$, $E_M = I - M_{10} \approx -0.008248$, and $E_S = I - S_{10} \approx -0.000110$.

(b) $f(x) = \sin x \quad \Rightarrow \quad \left|f^{(n)}(x)\right| \le 1$, so take $K = 1$ for all error estimates.

$|E_T| \le \dfrac{K(b-a)^3}{12n^2} = \dfrac{1(\pi - 0)^3}{12(10)^2} = \dfrac{\pi^3}{1200} \approx 0.025839$. $\quad |E_M| \le \dfrac{|E_T|}{2} = \dfrac{\pi^3}{2400} \approx 0.012919$.

$|E_S| \le \dfrac{K(b-a)^5}{180n^4} = \dfrac{1(\pi - 0)^5}{180(10)^4} = \dfrac{\pi^5}{1{,}800{,}000} \approx 0.000170$.

The actual error is about 64% of the error estimate in all three cases.

(c) $|E_T| \le 0.00001 \quad \Leftrightarrow \quad \dfrac{\pi^3}{12n^2} \le \dfrac{1}{10^5} \quad \Leftrightarrow \quad n^2 \ge \dfrac{10^5\pi^3}{12} \quad \Rightarrow \quad n \ge 508.3$. Take $n = 509$ for T_n.

$|E_M| \le 0.00001 \quad \Leftrightarrow \quad \dfrac{\pi^3}{24n^2} \le \dfrac{1}{10^5} \quad \Leftrightarrow \quad n^2 \ge \dfrac{10^5\pi^3}{24} \quad \Rightarrow \quad n \ge 359.4$. Take $n = 360$ for M_n.

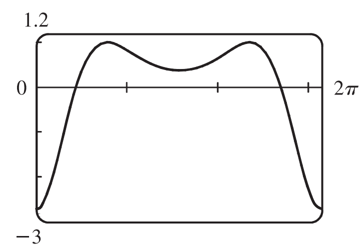
$|E_S| \le 0.00001 \quad \Leftrightarrow \quad \dfrac{\pi^5}{180n^4} \le \dfrac{1}{10^5} \quad \Leftrightarrow \quad n^4 \ge \dfrac{10^5\pi^5}{180} \quad \Rightarrow \quad n \ge 20.3$.

Take $n = 22$ for S_n (since n must be even).

20. From Example 7(b), we take $K = 76e$ to get $|E_S| \le \dfrac{76e(1)^5}{180n^4} \le 0.00001 \quad \Rightarrow \quad n^4 \ge \dfrac{76e}{180(0.00001)} \quad \Rightarrow \quad n \ge 18.4$.

Take $n = 20$ (since n must be even).

21. (a) Using a CAS, we differentiate $f(x) = e^{\cos x}$ twice, and find that $f''(x) = e^{\cos x}(\sin^2 x - \cos x)$. From the graph, we see that the maximum value of $|f''(x)|$ occurs at the endpoints of the interval $[0, 2\pi]$. Since $f''(0) = -e$, we can use $K = e$ or $K = 2.8$.



(b) A CAS gives $M_{10} \approx 7.954926518$. (In Maple, use `student[middlesum]`.)

(c) Using Theorem 3 for the Midpoint Rule, with $K = e$, we get $|E_M| \le \dfrac{e(2\pi - 0)^3}{24 \cdot 10^2} \approx 0.280945995$.

With $K = 2.8$, we get $|E_M| \le \dfrac{2.8(2\pi - 0)^3}{24 \cdot 10^2} = 0.289391916$.

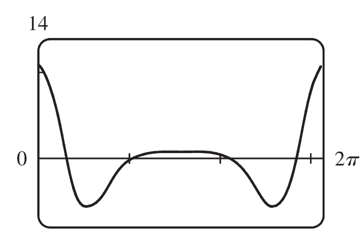
(d) A CAS gives $I \approx 7.954926521$.

(e) The actual error is only about 3×10^{-9}, much less than the estimate in part (c).

(f) We use the CAS to differentiate twice more, and then graph

$f^{(4)}(x) = e^{\cos x}(\sin^4 x - 6\sin^2 x\, \cos x + 3 - 7\sin^2 x + \cos x)$.

From the graph, we see that the maximum value of $\left|f^{(4)}(x)\right|$ occurs at the endpoints of the interval $[0, 2\pi]$. Since $f^{(4)}(0) = 4e$, we can use $K = 4e$ or $K = 10.9$.



(g) A CAS gives $S_{10} \approx 7.953789422$. (In Maple, use `student[simpson]`.)

(h) Using Theorem 4 with $K = 4e$, we get $|E_S| \le \dfrac{4e(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059153618$.

With $K = 10.9$, we get $|E_S| \le \dfrac{10.9(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059299814$.

(i) The actual error is about $7.954926521 - 7.953789422 \approx 0.00114$. This is quite a bit smaller than the estimate in part (h), though the difference is not nearly as great as it was in the case of the Midpoint Rule.

(j) To ensure that $|E_S| \le 0.0001$, we use Theorem 4: $|E_S| \le \dfrac{4e(2\pi)^5}{180 \cdot n^4} \le 0.0001 \quad\Rightarrow\quad \dfrac{4e(2\pi)^5}{180 \cdot 0.0001} \le n^4 \quad\Rightarrow$

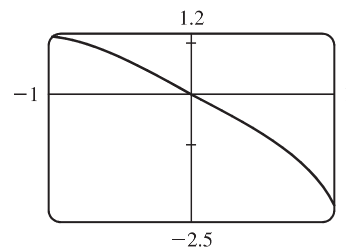
$n^4 \ge 5{,}915{,}362 \quad\Leftrightarrow\quad n \ge 49.3$. So we must take $n \ge 50$ to ensure that $|I - S_n| \le 0.0001$.

($K = 10.9$ leads to the same value of n.)

22. (a) Using the CAS, we differentiate $f(x) = \sqrt{4 - x^3}$ twice, and find

that $f''(x) = -\dfrac{9x^4}{4(4 - x^3)^{3/2}} - \dfrac{3x}{(4 - x^3)^{1/2}}$.

From the graph, we see that $|f''(x)| < 2.2$ on $[-1, 1]$.



(b) A CAS gives $M_{10} \approx 3.995804152$. (In Maple, use `student[middlesum]`.)

(c) Using Theorem 3 for the Midpoint Rule, with $K = 2.2$, we get $|E_M| \le \dfrac{2.2\,[1 - (-1)]^3}{24 \cdot 10^2} \approx 0.00733$.

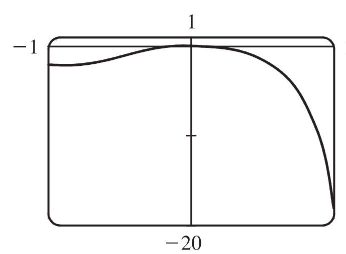
(d) A CAS gives $I \approx 3.995487677$.

(e) The actual error is about -0.0003165, much less than the estimate in part (c).

(f) We use the CAS to differentiate twice more, and then graph

$f^{(4)}(x) = \dfrac{9}{16}\dfrac{x^2(x^6 - 224x^3 - 1280)}{(4 - x^3)^{7/2}}$.

From the graph, we see that $\left|f^{(4)}(x)\right| < 18.1$ on $[-1, 1]$.



(g) A CAS gives $S_{10} \approx 3.995449790$. (In Maple, use `student[simpson]`.)

(h) Using Theorem 4 with $K = 18.1$, we get $|E_S| \le \dfrac{18.1\,[1 - (-1)]^5}{180 \cdot 10^4} \approx 0.000322$.

(i) The actual error is about $3.995487677 - 3.995449790 \approx 0.0000379$. This is quite a bit smaller than the estimate in part (h).

(j) To ensure that $|E_S| \le 0.0001$, we use Theorem 4: $|E_S| \le \dfrac{18.1(2)^5}{180 \cdot n^4} \le 0.0001 \quad\Rightarrow\quad \dfrac{18.1(2)^5}{180 \cdot 0.0001} \le n^4 \quad\Rightarrow$

$n^4 \ge 32{,}178 \quad\Rightarrow n \ge 13.4$. So we must take $n \ge 14$ to ensure that $|I - S_n| \le 0.0001$.

23. $I = \int_0^1 xe^x\,dx = [(x-1)e^x]_0^1$ [parts or Formula 96] $= 0 - (-1) = 1$, $f(x) = xe^x$, $\Delta x = 1/n$

$n = 5$:
$$L_5 = \tfrac{1}{5}[f(0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)] \approx 0.742943$$
$$R_5 = \tfrac{1}{5}[f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)] \approx 1.286599$$
$$T_5 = \tfrac{1}{5 \cdot 2}[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1)] \approx 1.014771$$
$$M_5 = \tfrac{1}{5}[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)] \approx 0.992621$$

$$E_L = I - L_5 \approx 1 - 0.742943 = 0.257057$$
$$E_R \approx 1 - 1.286599 = -0.286599$$
$$E_T \approx 1 - 1.014771 = -0.014771$$
$$E_M \approx 1 - 0.992621 = 0.007379$$

$n = 10$:
$$L_{10} = \tfrac{1}{10}[f(0) + f(0.1) + f(0.2) + \cdots + f(0.9)] \approx 0.867782$$
$$R_{10} = \tfrac{1}{10}[f(0.1) + f(0.2) + \cdots + f(0.9) + f(1)] \approx 1.139610$$
$$T_{10} = \tfrac{1}{10 \cdot 2}\{f(0) + 2[f(0.1) + f(0.2) + \cdots + f(0.9)] + f(1)\} \approx 1.003696$$
$$M_{10} = \tfrac{1}{10}[f(0.05) + f(0.15) + \cdots + f(0.85) + f(0.95)] \approx 0.998152$$

$$E_L = I - L_{10} \approx 1 - 0.867782 = 0.132218$$
$$E_R \approx 1 - 1.139610 = -0.139610$$
$$E_T \approx 1 - 1.003696 = -0.003696$$
$$E_M \approx 1 - 0.998152 = 0.001848$$

$n = 20$:
$$L_{20} = \tfrac{1}{20}[f(0) + f(0.05) + f(0.10) + \cdots + f(0.95)] \approx 0.932967$$
$$R_{20} = \tfrac{1}{20}[f(0.05) + f(0.10) + \cdots + f(0.95) + f(1)] \approx 1.068881$$
$$T_{20} = \tfrac{1}{20 \cdot 2}\{f(0) + 2[f(0.05) + f(0.10) + \cdots + f(0.95)] + f(1)\} \approx 1.000924$$
$$M_{20} = \tfrac{1}{20}[f(0.025) + f(0.075) + f(0.125) + \cdots + f(0.975)] \approx 0.999538$$

$$E_L = I - L_{20} \approx 1 - 0.932967 = 0.067033$$
$$E_R \approx 1 - 1.068881 = -0.068881$$
$$E_T \approx 1 - 1.000924 = -0.000924$$
$$E_M \approx 1 - 0.999538 = 0.000462$$

n	L_n	R_n	T_n	M_n
5	0.742943	1.286599	1.014771	0.992621
10	0.867782	1.139610	1.003696	0.998152
20	0.932967	1.068881	1.000924	0.999538

n	E_L	E_R	E_T	E_M
5	0.257057	−0.286599	−0.014771	0.007379
10	0.132218	−0.139610	−0.003696	0.001848
20	0.067033	−0.068881	−0.000924	0.000462

Observations:

1. E_L and E_R are always opposite in sign, as are E_T and E_M.
2. As n is doubled, E_L and E_R are decreased by about a factor of 2, and E_T and E_M are decreased by a factor of about 4.
3. The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
4. All the approximations become more accurate as the value of n increases.
5. The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

24. $I = \int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2} - (-1) = \frac{1}{2}, f(x) = \frac{1}{x^2}, \Delta x = \frac{1}{n}$

$n = 5$:

$L_5 = \frac{1}{5}[f(1) + f(1.2) + f(1.4) + f(1.6) + f(1.8)] \approx 0.580783$

$R_5 = \frac{1}{5}[f(1.2) + f(1.4) + f(1.6) + f(1.8) + f(2)] \approx 0.430783$

$T_5 = \frac{1}{5 \cdot 2}[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)] \approx 0.505783$

$M_5 = \frac{1}{5}[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)] \approx 0.497127$

$E_L = I - L_5 \approx \frac{1}{2} - 0.580783 = -0.080783$

$E_R \approx \frac{1}{2} - 0.430783 = 0.069217$

$E_T \approx \frac{1}{2} - 0.505783 = -0.005783$

$E_M \approx \frac{1}{2} - 0.497127 = 0.002873$

$n = 10$:

$L_{10} = \frac{1}{10}[f(1) + f(1.1) + f(1.2) + \cdots + f(1.9)] \approx 0.538955$

$R_{10} = \frac{1}{10}[f(1.1) + f(1.2) + \cdots + f(1.9) + f(2)] \approx 0.463955$

$T_{10} = \frac{1}{10 \cdot 2}\{f(1) + 2[f(1.1) + f(1.2) + \cdots + f(1.9)] + f(2)\} \approx 0.501455$

$M_{10} = \frac{1}{10}[f(1.05) + f(1.15) + \cdots + f(1.85) + f(1.95)] \approx 0.499274$

$E_L = I - L_{10} \approx \frac{1}{2} - 0.538955 = -0.038955$

$E_R \approx \frac{1}{2} - 0.463955 = 0.036049$

$E_T \approx \frac{1}{2} - 0.501455 = -0.001455$

$E_M \approx \frac{1}{2} - 0.499274 = 0.000726$

$n = 20$:

$L_{20} = \frac{1}{20}[f(1) + f(1.05) + f(1.10) + \cdots + f(1.95)] \approx 0.519114$

$R_{20} = \frac{1}{20}[f(1.05) + f(1.10) + \cdots + f(1.95) + f(2)] \approx 0.481614$

$T_{20} = \frac{1}{20 \cdot 2}\{f(1) + 2[f(1.05) + f(1.10) + \cdots + f(1.95)] + f(2)\} \approx 0.500364$

$M_{20} = \frac{1}{20}[f(1.025) + f(1.075) + f(1.125) + \cdots + f(1.975)] \approx 0.499818$

$E_L = I - L_{20} \approx \frac{1}{2} - 0.519114 = -0.019114$

$E_R \approx \frac{1}{2} - 0.481614 = 0.018386$

$E_T \approx \frac{1}{2} - 0.500364 = -0.000364$

$E_M \approx \frac{1}{2} - 0.499818 = 0.000182$

n	L_n	R_n	T_n	M_n
5	0.580783	0.430783	0.505783	0.497127
10	0.538955	0.463955	0.501455	0.499274
20	0.519114	0.481614	0.500364	0.499818

n	E_L	E_R	E_T	E_M
5	−0.080783	0.069217	−0.005783	0.002873
10	−0.038955	0.036049	−0.001455	0.000726
20	−0.019114	0.018386	−0.000364	0.000182

Observations:

1. E_L and E_R are always opposite in sign, as are E_T and E_M.
2. As n is doubled, E_L and E_R are decreased by about a factor of 2, and E_T and E_M are decreased by a factor of about 4.
3. The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
4. All the approximations become more accurate as the value of n increases.
5. The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

25. $I = \int_0^2 x^4\,dx = \left[\frac{1}{5}x^5\right]_0^2 = \frac{32}{5} - 0 = 6.4$, $f(x) = x^4$, $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

$n = 6$:

$$T_6 = \tfrac{2}{6\cdot 2}\left\{f(0) + 2\left[f\left(\tfrac{1}{3}\right) + f\left(\tfrac{2}{3}\right) + f\left(\tfrac{3}{3}\right) + f\left(\tfrac{4}{3}\right) + f\left(\tfrac{5}{3}\right)\right] + f(2)\right\} \approx 6.695473$$

$$M_6 = \tfrac{2}{6}\left[f\left(\tfrac{1}{6}\right) + f\left(\tfrac{3}{6}\right) + f\left(\tfrac{5}{6}\right) + f\left(\tfrac{7}{6}\right) + f\left(\tfrac{9}{6}\right) + f\left(\tfrac{11}{6}\right)\right] \approx 6.252572$$

$$S_6 = \tfrac{2}{6\cdot 3}\left[f(0) + 4f\left(\tfrac{1}{3}\right) + 2f\left(\tfrac{2}{3}\right) + 4f\left(\tfrac{3}{3}\right) + 2f\left(\tfrac{4}{3}\right) + 4f\left(\tfrac{5}{3}\right) + f(2)\right] \approx 6.403292$$

$$E_T = I - T_6 \approx 6.4 - 6.695473 = -0.295473$$

$$E_M \approx 6.4 - 6.252572 = 0.147428$$

$$E_S \approx 6.4 - 6.403292 = -0.003292$$

$n = 12$:

$$T_{12} = \tfrac{2}{12\cdot 2}\left\{f(0) + 2\left[f\left(\tfrac{1}{6}\right) + f\left(\tfrac{2}{6}\right) + f\left(\tfrac{3}{6}\right) + \cdots + f\left(\tfrac{11}{6}\right)\right] + f(2)\right\} \approx 6.474023$$

$$M_6 = \tfrac{2}{12}\left[f\left(\tfrac{1}{12}\right) + f\left(\tfrac{3}{12}\right) + f\left(\tfrac{5}{12}\right) + \cdots + f\left(\tfrac{23}{12}\right)\right] \approx 6.363008$$

$$S_6 = \tfrac{2}{12\cdot 3}\left[f(0) + 4f\left(\tfrac{1}{6}\right) + 2f\left(\tfrac{2}{6}\right) + 4f\left(\tfrac{3}{6}\right) + 2f\left(\tfrac{4}{6}\right) + \cdots + 4f\left(\tfrac{11}{6}\right) + f(2)\right] \approx 6.400206$$

$$E_T = I - T_{12} \approx 6.4 - 6.474023 = -0.074023$$

$$E_M \approx 6.4 - 6.363008 = 0.036992$$

$$E_S \approx 6.4 - 6.400206 = -0.000206$$

n	T_n	M_n	S_n
6	6.695473	6.252572	6.403292
12	6.474023	6.363008	6.400206

n	E_T	E_M	E_S
6	−0.295473	0.147428	−0.003292
12	−0.074023	0.036992	−0.000206

Observations:

1. E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.
2. The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and E_S seems to decrease by a factor of about 16 as n is doubled.

26. $I = \displaystyle\int_1^4 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_1^4 = 4 - 2 = 2$, $f(x) = \dfrac{1}{\sqrt{x}}$, $\Delta x = \dfrac{4-1}{n} = \dfrac{3}{n}$

$n = 6$:

$$T_6 = \tfrac{3}{6\cdot 2}\left\{f(1) + 2\left[f\left(\tfrac{3}{2}\right) + f\left(\tfrac{4}{2}\right) + f\left(\tfrac{5}{2}\right) + f\left(\tfrac{6}{2}\right) + f\left(\tfrac{7}{2}\right)\right] + f(4)\right\} \approx 2.008966$$

$$M_6 = \tfrac{3}{6}\left[f\left(\tfrac{5}{4}\right) + f\left(\tfrac{7}{4}\right) + f\left(\tfrac{9}{4}\right) + f\left(\tfrac{11}{4}\right) + f\left(\tfrac{13}{4}\right) + f\left(\tfrac{15}{4}\right)\right] \approx 1.995572$$

$$S_6 = \tfrac{3}{6\cdot 3}\left[f(1) + 4f\left(\tfrac{3}{2}\right) + 2f\left(\tfrac{4}{2}\right) + 4f\left(\tfrac{5}{2}\right) + 2f\left(\tfrac{6}{2}\right) + 4f\left(\tfrac{7}{2}\right) + f(4)\right] \approx 2.000469$$

$$E_T = I - T_6 \approx 2 - 2.008966 = -0.008966,$$

$$E_M \approx 2 - 1.995572 = 0.004428,$$

$$E_S \approx 2 - 2.000469 = -0.000469$$

$n = 12$:

$$T_{12} = \tfrac{3}{12\cdot 2}\left\{f(1) + 2\left[f\left(\tfrac{5}{4}\right) + f\left(\tfrac{6}{4}\right) + f\left(\tfrac{7}{4}\right) + \cdots + f\left(\tfrac{15}{4}\right)\right] + f(4)\right\} \approx 2.002269$$

$$M_{12} = \tfrac{3}{12}\left[f\left(\tfrac{9}{8}\right) + f\left(\tfrac{11}{8}\right) + f\left(\tfrac{13}{8}\right) + \cdots + f\left(\tfrac{31}{8}\right)\right] \approx 1.998869$$

$$S_{12} = \tfrac{3}{12\cdot 3}\left[f(1) + 4f\left(\tfrac{5}{4}\right) + 2f\left(\tfrac{6}{4}\right) + 4f\left(\tfrac{7}{4}\right) + 2f\left(\tfrac{8}{4}\right) + \cdots + 4f\left(\tfrac{15}{4}\right) + f(4)\right] \approx 2.000036$$

$$E_T = I - T_{12} \approx 2 - 2.002269 = -0.002269$$

$$E_M \approx 2 - 1.998869 = 0.001131$$

$$E_S \approx 2 - 2.000036 = -0.000036$$

[continued]

n	T_n	M_n	S_n
6	2.008966	1.995572	2.000469
12	2.002269	1.998869	2.000036

n	E_T	E_M	E_S
6	−0.008966	0.004428	−0.000469
12	−0.002269	0.001131	−0.000036

Observations:

1. E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.
2. The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and E_S seems to decrease by a factor of about 16 as n is doubled.

27. $\Delta x = (b-a)/n = (6-0)/6 = 1$

(a) $T_6 = \frac{\Delta x}{2}[f(0) + 2f(1) + 2f(2) + 2f(3) + 2f(4) + 2f(5) + f(6)]$

$\approx \frac{1}{2}[3 + 2(5) + 2(4) + 2(2) + 2(2.8) + 2(4) + 1]$

$= \frac{1}{2}(39.6) = 19.8$

(b) $M_6 = \Delta x[f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)]$

$\approx 1[4.5 + 4.7 + 2.6 + 2.2 + 3.4 + 3.2]$

$= 20.6$

(c) $S_6 = \frac{\Delta x}{3}[f(0) + 4f(1) + 2f(2) + 4f(3) + 2f(4) + 4f(5) + f(6)]$

$\approx \frac{1}{3}[3 + 4(5) + 2(4) + 4(2) + 2(2.8) + 4(4) + 1]$

$= \frac{1}{3}(61.6) = 20.5\overline{3}$

28. We use Simpson's Rule with $n = 10$ and $\Delta x = \frac{1}{2}$:

$$\begin{aligned}\text{distance} = \int_0^5 v(t)\,dt \approx S_{10} &= \tfrac{1}{2\cdot 3}[f(0) + 4f(0.5) + 2f(1) + \cdots + 4f(4.5) + f(5)] \\ &= \tfrac{1}{6}[0 + 4(4.67) + 2(7.34) + 4(8.86) + 2(9.73) + 4(10.22) \\ &\qquad + 2(10.51) + 4(10.67) + 2(10.76) + 4(10.81) + 10.81] \\ &= \tfrac{1}{6}(268.41) = 44.735 \text{ m}\end{aligned}$$

29. By the Net Change Theorem, the increase in velocity is equal to $\int_0^6 a(t)\,dt$. We use Simpson's Rule with $n = 6$ and $\Delta t = (6-0)/6 = 1$ to estimate this integral:

$$\begin{aligned}\int_0^6 a(t)\,dt \approx S_6 &= \tfrac{1}{3}[a(0) + 4a(1) + 2a(2) + 4a(3) + 2a(4) + 4a(5) + a(6)] \\ &\approx \tfrac{1}{3}[0 + 4(0.5) + 2(4.1) + 4(9.8) + 2(12.9) + 4(9.5) + 0] = \tfrac{1}{3}(113.2) = 37.7\overline{3} \text{ ft/s}\end{aligned}$$

30. By the Net Change Theorem, the total amount of water that leaked out during the first six hours is equal to $\int_0^6 r(t)\,dt$. We use Simpson's Rule with $n = 6$ and $\Delta t = \frac{6-0}{6} = 1$ to estimate this integral:

$$\begin{aligned}\int_0^6 r(t)\,dt \approx S_6 &= \tfrac{1}{3}[r(0) + 4r(1) + 2r(2) + 4r(3) + 2r(4) + 4r(5) + r(6)] \\ &\approx \tfrac{1}{3}[4 + 4(3) + 2(2.4) + 4(1.9) + 2(1.4) + 4(1.1) + 1] = \tfrac{1}{3}(36.6) = 12.2 \text{ liters}\end{aligned}$$

31. By the Net Change Theorem, the energy used is equal to $\int_0^6 P(t)\,dt$. We use Simpson's Rule with $n = 12$ and $\Delta t = \frac{6-0}{12} = \frac{1}{2}$ to estimate this integral:

$$\begin{aligned}\int_0^6 P(t)\,dt \approx S_{12} &= \tfrac{1/2}{3}[P(0)+4P(0.5)+2P(1)+4P(1.5)+2P(2)+4P(2.5)+2P(3)\\ &\qquad +4P(3.5)+2P(4)+4P(4.5)+2P(5)+4P(5.5)+P(6)]\\ &= \tfrac{1}{6}[1814+4(1735)+2(1686)+4(1646)+2(1637)+4(1609)+2(1604)\\ &\qquad +4(1611)+2(1621)+4(1666)+2(1745)+4(1886)+2052]\\ &= \tfrac{1}{6}(61{,}064) = 10{,}177.\overline{3} \text{ megawatt-hours}\end{aligned}$$

32. By the Net Change Theorem, the total amount of data transmitted is equal to $\int_0^8 D(t)\,dt \times 3600$ [since $D(t)$ is measured in megabits per second and t is in hours]. We use Simpson's Rule with $n = 8$ and $\Delta t = (8-0)/8 = 1$ to estimate this integral:

$$\begin{aligned}\int_0^8 D(t)\,dt \approx S_8 &= \tfrac{1}{3}[D(0)+4D(1)+2D(2)+4D(3)+2D(4)+4D(5)+2D(6)+4D(7)+D(8)]\\ &\approx \tfrac{1}{3}[0.35+4(0.32)+2(0.41)+4(0.50)+2(0.51)+4(0.56)+2(0.56)+4(0.83)+0.88]\\ &= \tfrac{1}{3}(13.03) = 4.34\overline{3}\end{aligned}$$

Now multiply by 3600 to obtain 15,636 megabits.

33. (a) We are given the function values at the endpoints of 8 intervals of length 0.4, so we'll use the Midpoint Rule with $n = 8/2 = 4$ and $\Delta x = (3.2-0)/4 = 0.8$.

$$\begin{aligned}\int_0^{3.2} f(x)\,dx \approx M_4 &= 0.8[f(0.4)+f(1.2)+f(2.0)+f(2.8)] = 0.8[6.5+6.4+7.6+8.8]\\ &= 0.8(29.3) = 23.44\end{aligned}$$

(b) $-4 \le f''(x) \le 1 \quad\Rightarrow\quad |f''(x)| \le 4$, so use $K = 4$, $a = 0$, $b = 3.2$, and $n = 4$ in Theorem 3.

So $|E_M| \le \dfrac{4(3.2-0)^3}{24(4)^2} = \dfrac{128}{375} = 0.341\overline{3}$.

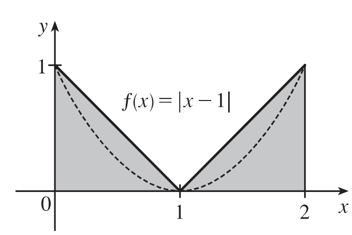
34. Using Simpson's Rule with $n = 10$, $\Delta x = \frac{\pi/2}{10}$, $L = 1$, $\theta_0 = \frac{42\pi}{180}$ radians, $g = 9.8\text{ m/s}^2$, $k^2 = \sin^2\!\left(\frac{1}{2}\theta_0\right)$, and $f(x) = 1/\sqrt{1-k^2\sin^2 x}$, we get

$$\begin{aligned}T &= 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-k^2\sin^2 x}} \approx 4\sqrt{\frac{L}{g}}\,S_{10}\\ &= 4\sqrt{\tfrac{1}{9.8}}\left(\tfrac{\pi/2}{10\cdot 3}\right)\left[f(0)+4f\left(\tfrac{\pi}{20}\right)+2f\left(\tfrac{2\pi}{20}\right)+\cdots+4f\left(\tfrac{9\pi}{20}\right)+f\left(\tfrac{\pi}{2}\right)\right] \approx 2.07665\end{aligned}$$

35. $I(\theta) = \dfrac{N^2\sin^2 k}{k^2}$, where $k = \dfrac{\pi N d\sin\theta}{\lambda}$, $N = 10{,}000$, $d = 10^{-4}$, and $\lambda = 632.8\times 10^{-9}$. So $I(\theta) = \dfrac{(10^4)^2\sin^2 k}{k^2}$, where $k = \dfrac{\pi(10^4)(10^{-4})\sin\theta}{632.8\times 10^{-9}}$. Now $n = 10$ and $\Delta\theta = \dfrac{10^{-6}-(-10^{-6})}{10} = 2\times 10^{-7}$, so

$M_{10} = 2\times 10^{-7}[I(-0.0000009)+I(-0.0000007)+\cdots+I(0.0000009)] \approx 59.4$.

36. Consider the function $f(x) = |x-1|$, $0 \le x \le 2$. The area $\int_0^2 f(x)\,dx$ is exactly 1. So is the right endpoint approximation: $R_2 = f(1)\,\Delta x + f(2)\,\Delta x = 0\cdot 1 + 1\cdot 1 = 1$. But Simpson's Rule approximates f with the parabola $y = (x-1)^2$, shown dashed, and $S_2 = \dfrac{\Delta x}{3}[f(0)+4f(1)+f(2)] = \dfrac{1}{3}[1+4\cdot 0+1] = \dfrac{2}{3}$.

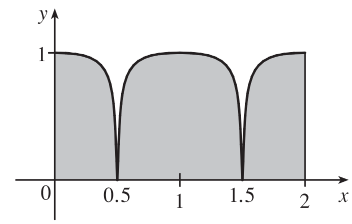


37. Consider the function f whose graph is shown. The area $\int_0^2 f(x)\,dx$ is close to 2. The Trapezoidal Rule gives

$T_2 = \frac{2-0}{2\cdot 2}\,[f(0) + 2f(1) + f(2)] = \frac{1}{2}\,[1 + 2\cdot 1 + 1] = 2.$

The Midpoint Rule gives $M_2 = \frac{2-0}{2}\,[f(0.5) + f(1.5)] = 1[0+0] = 0,$

so the Trapezoidal Rule is more accurate.



38. $f(x) = \cos(\pi x)$, $\Delta x = \frac{20-0}{10} = 2 \quad \Rightarrow$

$$\begin{aligned} T_{10} &= \tfrac{2}{2}\{f(0) + 2[f(2) + f(4) + \cdots + f(18)] + f(20)\} = 1[\cos 0 + 2(\cos 2\pi + \cos 4\pi + \cdots + \cos 18\pi) + \cos 20\pi] \\ &= 1 + 2(1+1+1+1+1+1+1+1+1) + 1 = 20 \end{aligned}$$

The actual value is $\int_0^{20} \cos(\pi x)\,dx = \frac{1}{\pi}\big[\sin \pi x\big]_0^{20} = \frac{1}{\pi}(\sin 20\pi - \sin 0) = 0$. The discrepancy is due to the fact that the function is sampled only at points of the form $2n$, where its value is $f(2n) = \cos(2n\pi) = 1$.

39. Since the Trapezoidal and Midpoint approximations on the interval $[a,b]$ are the sums of the Trapezoidal and Midpoint approximations on the subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, we can focus our attention on one such interval. The condition $f''(x) < 0$ for $a \le x \le b$ means that the graph of f is concave down as in Figure 5. In that figure, T_n is the area of the trapezoid $AQRD$, $\int_a^b f(x)\,dx$ is the area of the region $AQPRD$, and M_n is the area of the trapezoid $ABCD$, so $T_n < \int_a^b f(x)\,dx < M_n$. In general, the condition $f'' < 0$ implies that the graph of f on $[a,b]$ lies above the chord joining the points $(a, f(a))$ and $(b, f(b))$. Thus, $\int_a^b f(x)\,dx > T_n$. Since M_n is the area under a tangent to the graph, and since $f'' < 0$ implies that the tangent lies above the graph, we also have $M_n > \int_a^b f(x)\,dx$. Thus, $T_n < \int_a^b f(x)\,dx < M_n$.

40. Let f be a polynomial of degree ≤ 3; say $f(x) = Ax^3 + Bx^2 + Cx + D$. It will suffice to show that Simpson's estimate is exact when there are two subintervals ($n = 2$), because for a larger even number of subintervals the sum of exact estimates is exact. As in the derivation of Simpson's Rule, we can assume that $x_0 = -h$, $x_1 = 0$, and $x_2 = h$. Then Simpson's approximation is

$$\begin{aligned} \int_{-h}^{h} f(x)\,dx &\approx \tfrac{1}{3}h[f(-h) + 4f(0) + f(h)] = \tfrac{1}{3}h\big[\big(-Ah^3 + Bh^2 - Ch + D\big) + 4D + \big(Ah^3 + Bh^2 + Ch + D\big)\big] \\ &= \tfrac{1}{3}h[2Bh^2 + 6D] = \tfrac{2}{3}Bh^3 + 2Dh \end{aligned}$$

The exact value of the integral is

$$\begin{aligned} \int_{-h}^{h} (Ax^3 + Bx^2 + Cx + D)\,dx &= 2\int_0^h (Bx^2 + D)\,dx \qquad \text{[by Theorem 5.5.6(a) and (b)]} \\ &= 2\big[\tfrac{1}{3}Bx^3 + Dx\big]_0^h = \tfrac{2}{3}Bh^3 + 2Dh \end{aligned}$$

Thus, Simpson's Rule is exact.

41. $T_n = \frac{1}{2}\,\Delta x\,[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$ and

$M_n = \Delta x\,[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_{n-1}) + f(\overline{x}_n)]$, where $\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$. Now

$T_{2n} = \frac{1}{2}\big(\frac{1}{2}\Delta x\big)[f(x_0) + 2f(\overline{x}_1) + 2f(x_1) + 2f(\overline{x}_2) + 2f(x_2) + \cdots + 2f(\overline{x}_{n-1}) + 2f(x_{n-1}) + 2f(\overline{x}_n) + f(x_n)]$ so

$$\begin{aligned} \tfrac{1}{2}(T_n + M_n) &= \tfrac{1}{2}T_n + \tfrac{1}{2}M_n \\ &= \tfrac{1}{4}\Delta x[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)] + \tfrac{1}{4}\Delta x[2f(\overline{x}_1) + 2f(\overline{x}_2) + \cdots + 2f(\overline{x}_{n-1}) + 2f(\overline{x}_n)] \\ &= T_{2n} \end{aligned}$$

42. $T_n = \dfrac{\Delta x}{2}\left[f(x_0) + 2\sum\limits_{i=1}^{n-1} f(x_i) + f(x_n)\right]$ and $M_n = \Delta x \sum\limits_{i=1}^{n} f\left(x_i - \dfrac{\Delta x}{2}\right)$, so

$$\tfrac{1}{3}T_n + \tfrac{2}{3}M_n = \tfrac{1}{3}(T_n + 2M_n) = \frac{\Delta x}{3\cdot 2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) + 4\sum_{i=1}^{n} f\left(x_i - \frac{\Delta x}{2}\right)\right]$$

where $\Delta x = \dfrac{b-a}{n}$. Let $\delta x = \dfrac{b-a}{2n}$. Then $\Delta x = 2\delta x$, so

$$\begin{aligned}\tfrac{1}{3}T_n + \tfrac{2}{3}M_n &= \frac{\delta x}{3}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) + 4\sum_{i=1}^{n} f(x_i - \delta x)\right] \\ &= \tfrac{1}{3}\delta x[f(x_0) + 4f(x_1 - \delta x) + 2f(x_1) + 4f(x_2 - \delta x) \\ &\qquad + 2f(x_2) + \cdots + 2f(x_{n-1}) + 4f(x_n - \delta x) + f(x_n)]\end{aligned}$$

Since $x_0, x_1 - \delta x, x_1, x_2 - \delta x, x_2, \ldots, x_{n-1}, x_n - \delta x, x_n$ are the subinterval endpoints for S_{2n}, and since $\delta x = \dfrac{b-a}{2n}$ is the width of the subintervals for S_{2n}, the last expression for $\frac{1}{3}T_n + \frac{2}{3}M_n$ is the usual expression for S_{2n}. Therefore, $\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$.

5.10 Improper Integrals

1. (a) Since $\int_1^\infty x^4 e^{-x^4}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(b) Since $y = \sec x$ has an infinite discontinuity at $x = \frac{\pi}{2}$, $\int_0^{\pi/2} \sec x\,dx$ is a Type II improper integral.

(c) Since $y = \dfrac{x}{(x-2)(x-3)}$ has an infinite discontinuity at $x = 2$, $\displaystyle\int_0^2 \frac{x}{x^2 - 5x + 6}\,dx$ is a Type II improper integral.

(d) Since $\displaystyle\int_{-\infty}^0 \frac{1}{x^2+5}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

2. (a) Since $y = \dfrac{1}{2x-1}$ is defined and continuous on $[1, 2]$, $\displaystyle\int_1^2 \frac{1}{2x-1}\,dx$ is proper.

(b) Since $y = \dfrac{1}{2x-1}$ has an infinite discontinuity at $x = \frac{1}{2}$, $\displaystyle\int_0^1 \frac{1}{2x-1}\,dx$ is a Type II improper integral.

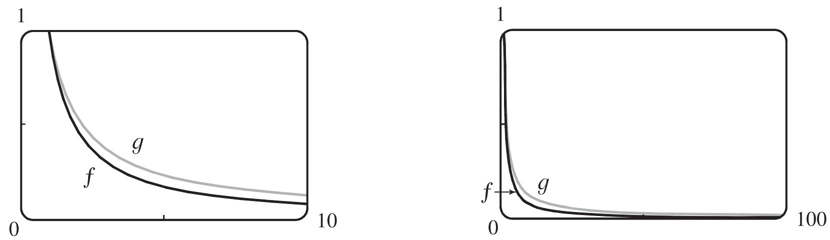
(c) Since $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{1+x^2}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(d) Since $y = \ln(x-1)$ has an infinite discontinuity at $x = 1$, $\int_1^2 \ln(x-1)dx$ is a Type II improper integral.

3. The area under the graph of $y = 1/x^3 = x^{-3}$ between $x = 1$ and $x = t$ is

$A(t) = \int_1^t x^{-3}\,dx = \left[-\frac{1}{2}x^{-2}\right]_1^t = -\frac{1}{2}t^{-2} - \left(-\frac{1}{2}\right) = \frac{1}{2} - 1/(2t^2)$. So the area for $1 \le x \le 10$ is $A(10) = 0.5 - 0.005 = 0.495$, the area for $1 \le x \le 100$ is $A(100) = 0.5 - 0.00005 = 0.49995$, and the area for $1 \le x \le 1000$ is $A(1000) = 0.5 - 0.0000005 = 0.4999995$. The total area under the curve for $x \ge 1$ is $\lim\limits_{t\to\infty} A(t) = \lim\limits_{t\to\infty}\left[\frac{1}{2} - 1/(2t^2)\right] = \frac{1}{2}$.

4. (a)



(b) The area under the graph of f from $x = 1$ to $x = t$ is

$$F(t) = \int_1^t f(x)\,dx = \int_1^t x^{-1.1}\,dx = \left[-\tfrac{1}{0.1}x^{-0.1}\right]_1^t$$
$$= -10(t^{-0.1} - 1) = 10(1 - t^{-0.1})$$

and the area under the graph of g is

$$G(t) = \int_1^t g(x)\,dx = \int_1^t x^{-0.9}\,dx = \left[\tfrac{1}{0.1}x^{0.1}\right]_1^t = 10(t^{0.1} - 1).$$

t	$F(t)$	$G(t)$
10	2.06	2.59
100	3.69	5.85
10^4	6.02	15.12
10^6	7.49	29.81
10^{10}	9	90
10^{20}	9.9	990

(c) The total area under the graph of f is $\lim\limits_{t\to\infty} F(t) = \lim\limits_{t\to\infty} 10(1 - t^{-0.1}) = 10$.

The total area under the graph of g does not exist, since $\lim\limits_{t\to\infty} G(t) = \lim\limits_{t\to\infty} 10(t^{0.1} - 1) = \infty$.

5. $$\int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx = \lim_{t\to\infty}\int_3^t (x-2)^{-3/2}\,dx = \lim_{t\to\infty}\left[-2\,(x-2)^{-1/2}\right]_3^t \qquad [u = x-2,\ du = dx]$$
$$= \lim_{t\to\infty}\left(\frac{-2}{\sqrt{t-2}} + \frac{2}{\sqrt{1}}\right) = 0 + 2 = 2. \qquad \text{Convergent}$$

6. $$\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx = \lim_{t\to\infty}\int_0^t (1+x)^{-1/4}\,dx = \lim_{t\to\infty}\left[\tfrac{4}{3}(1+x)^{3/4}\right]_0^t \qquad [u = 1+x,\ du = dx]$$
$$= \lim_{t\to\infty}\left[\tfrac{4}{3}(1+t)^{3/4} - \tfrac{4}{3}\right] = \infty. \qquad \text{Divergent}$$

7. $$\int_{-\infty}^{-1} \frac{1}{\sqrt{2-w}}\,dw = \lim_{t\to-\infty}\int_t^{-1} \frac{1}{\sqrt{2-w}}\,dw = \lim_{t\to-\infty}\left[-2\sqrt{2-w}\right]_t^{-1} \qquad [u = 2-w,\ du = -dw]$$
$$= \lim_{t\to-\infty}\left[-2\sqrt{3} + 2\sqrt{2-t}\right] = \infty. \qquad \text{Divergent}$$

8. $$\int_0^\infty \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty}\frac{1}{2}\left[\frac{-1}{x^2+2}\right]_0^t = \frac{1}{2}\lim_{t\to\infty}\left(\frac{-1}{t^2+2} + \frac{1}{2}\right)$$
$$= \tfrac{1}{2}\left(0 + \tfrac{1}{2}\right) = \tfrac{1}{4}. \qquad \text{Convergent}$$

9. $\int_4^\infty e^{-y/2}\,dy = \lim\limits_{t\to\infty}\int_4^t e^{-y/2}\,dy = \lim\limits_{t\to\infty}\left[-2e^{-y/2}\right]_4^t = \lim\limits_{t\to\infty}(-2e^{-t/2} + 2e^{-2}) = 0 + 2e^{-2} = 2e^{-2}$.

Convergent

10. $\int_{-\infty}^{-1} e^{-2t}\,dt = \lim\limits_{x\to-\infty}\int_x^{-1} e^{-2t}\,dt = \lim\limits_{x\to-\infty}\left[-\tfrac{1}{2}e^{-2t}\right]_x^{-1} = \lim\limits_{x\to-\infty}\left[-\tfrac{1}{2}e^2 + \tfrac{1}{2}e^{-2x}\right] = \infty.$ Divergent

11. $\int_{2\pi}^\infty \sin\theta\,d\theta = \lim\limits_{t\to\infty}\int_{2\pi}^t \sin\theta\,d\theta = \lim\limits_{t\to\infty}\left[-\cos\theta\right]_{2\pi}^t = \lim\limits_{t\to\infty}(-\cos t + 1)$. This limit does not exist, so the integral is divergent. Divergent

12. $I = \int_{-\infty}^{\infty} (y^3 - 3y^2)\,dy = I_1 + I_2 = \int_{-\infty}^{0} (y^3 - 3y^2)\,dy + \int_{0}^{\infty} (y^3 - 3y^2)\,dy$, but

$I_1 = \lim\limits_{t \to -\infty} \left[\frac{1}{4}y^4 - y^3\right]_t^0 = \lim\limits_{t \to -\infty} (t^3 - \frac{1}{4}t^4) = -\infty$. Since I_1 is divergent, I is divergent,

and there is no need to evaluate I_2. Divergent

13. $\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx.$

$\int_{-\infty}^{0} xe^{-x^2}\,dx = \lim\limits_{t \to -\infty} \left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_t^0 = \lim\limits_{t \to -\infty} \left(-\frac{1}{2}\right)\left(1 - e^{-t^2}\right) = -\frac{1}{2} \cdot 1 = -\frac{1}{2}$, and

$\int_{0}^{\infty} xe^{-x^2}\,dx = \lim\limits_{t \to \infty} \left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_0^t = \lim\limits_{t \to \infty} \left(-\frac{1}{2}\right)\left(e^{-t^2} - 1\right) = -\frac{1}{2} \cdot (-1) = \frac{1}{2}.$

Therefore, $\int_{-\infty}^{\infty} xe^{-x^2}\,dx = -\frac{1}{2} + \frac{1}{2} = 0.$ Convergent

14. $$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{t \to \infty} \int_1^t \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{t \to \infty} \int_1^{\sqrt{t}} e^{-u}(2\,du) \qquad \begin{bmatrix} u = \sqrt{x}, \\ du = dx/(2\sqrt{x}) \end{bmatrix}$$

$$= 2\lim_{t \to \infty} \left[-e^{-u}\right]_1^{\sqrt{t}} = 2\lim_{t \to \infty} \left(-e^{-\sqrt{t}} + e^{-1}\right) = 2(0 + e^{-1}) = 2e^{-1}.$$ Convergent

15. $$\int_1^\infty \frac{x+1}{x^2+2x}\,dx = \lim_{t \to \infty} \int_1^t \frac{\frac{1}{2}(2x+2)}{x^2+2x}\,dx = \frac{1}{2}\lim_{t \to \infty} \left[\ln(x^2+2x)\right]_1^t = \frac{1}{2}\lim_{t \to \infty} \left[\ln(t^2+2t) - \ln 3\right] = \infty.$$

Divergent

16. $I = \int_{-\infty}^{\infty} \cos \pi t\,dt = I_1 + I_2 = \int_{-\infty}^{0} \cos \pi t\,dt + \int_{0}^{\infty} \cos \pi t\,dt$, but $I_1 = \lim\limits_{s \to -\infty} \left[\dfrac{1}{\pi}\sin \pi t\right]_s^0 = \lim\limits_{s \to -\infty} \left(-\dfrac{1}{\pi}\sin \pi t\right)$ and

this limit does not exist. Since I_1 is divergent, I is divergent, and there is no need to evaluate I_2. Divergent

17. $$\int_0^\infty se^{-5s}\,ds = \lim_{t \to \infty} \int_0^t se^{-5s}\,ds = \lim_{t \to \infty} \left[-\tfrac{1}{5}se^{-5s} - \tfrac{1}{25}e^{-5s}\right] \qquad \begin{bmatrix} \text{by integration by} \\ \text{parts with } u = s \end{bmatrix}$$

$$= \lim_{t \to \infty} \left(-\tfrac{1}{5}te^{-5t} - \tfrac{1}{25}e^{-5t} + \tfrac{1}{25}\right) = 0 - 0 + \tfrac{1}{25} \qquad [\text{by l'Hospital's Rule}]$$

$$= \tfrac{1}{25}.$$ Convergent

18. $$\int_{-\infty}^6 re^{r/3}\,dr = \lim_{t \to -\infty} \int_t^6 re^{r/3}\,dr = \lim_{t \to -\infty} \left[3re^{r/3} - 9e^{r/3}\right]_t^6 \qquad \begin{bmatrix} \text{by integration by} \\ \text{parts with } u = r \end{bmatrix}$$

$$= \lim_{t \to -\infty} (18e^2 - 9e^2 - 3te^{t/3} + 9e^{t/3}) = 9e^2 - 0 + 0 \qquad [\text{by l'Hospital's Rule}]$$

$$= 9e^2.$$ Convergent

19. $$\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t \to \infty} \left[\frac{(\ln x)^2}{2}\right]_1^t \qquad \begin{bmatrix} \text{by substitution with} \\ u = \ln x,\ du = dx/x \end{bmatrix} \qquad = \lim_{t \to \infty} \frac{(\ln t)^2}{2} = \infty.$$ Divergent

20. $I = \int_{-\infty}^{\infty} x^3e^{-x^4}\,dx = I_1 + I_2 = \int_{-\infty}^{0} x^3e^{-x^4}\,dx + \int_{0}^{\infty} x^3e^{-x^4}\,dx$. Now

$$I_2 = \lim_{t \to \infty} \int_0^t x^3e^{-x^4}\,dx = \lim_{t \to \infty} \int_0^{t^4} e^{-u}\left(\tfrac{1}{4}\,du\right) \qquad \begin{bmatrix} u = x^4, \\ du = 4x^3\,dx \end{bmatrix}$$

$$= \tfrac{1}{4}\lim_{t \to \infty} \left[-e^{-u}\right]_0^{t^4} = \tfrac{1}{4}\lim_{t \to \infty} \left(-e^{-t^4} + 1\right) = \tfrac{1}{4}(0 + 1) = \tfrac{1}{4}.$$

Since $f(x) = x^3e^{-x^4}$ is an odd function, $I_1 = -\frac{1}{4}$, and hence, $I = 0$. Convergent

21. $\displaystyle\int_{-\infty}^{\infty}\frac{x^2}{9+x^6}\,dx=\int_{-\infty}^{0}\frac{x^2}{9+x^6}\,dx+\int_{0}^{\infty}\frac{x^2}{9+x^6}\,dx=2\int_{0}^{\infty}\frac{x^2}{9+x^6}\,dx$ [since the integrand is even].

Now $\displaystyle\int\frac{x^2\,dx}{9+x^6}\;\left[\begin{array}{c}u=x^3\\ du=3x^2\,dx\end{array}\right]=\int\frac{\frac{1}{3}\,du}{9+u^2}\;\left[\begin{array}{c}u=3v\\ du=3\,dv\end{array}\right]=\int\frac{\frac{1}{3}(3\,dv)}{9+9v^2}=\frac{1}{9}\int\frac{dv}{1+v^2}$

$$=\frac{1}{9}\tan^{-1}v+C=\frac{1}{9}\tan^{-1}\left(\frac{u}{3}\right)+C=\frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right)+C,$$

so $\displaystyle 2\int_{0}^{\infty}\frac{x^2}{9+x^6}\,dx=2\lim_{t\to\infty}\int_{0}^{t}\frac{x^2}{9+x^6}\,dx=2\lim_{t\to\infty}\left[\frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right)\right]_0^t=2\lim_{t\to\infty}\frac{1}{9}\tan^{-1}\left(\frac{t^3}{3}\right)=\frac{2}{9}\cdot\frac{\pi}{2}=\frac{\pi}{9}.$

Convergent

22. Integrate by parts with $u=\ln x$, $dv=dx/x^3\;\Rightarrow\;du=dx/x$, $v=-1/(2x^2)$.

$$\int_{1}^{\infty}\frac{\ln x}{x^3}\,dx=\lim_{t\to\infty}\int_{1}^{t}\frac{\ln x}{x^3}\,dx=\lim_{t\to\infty}\left(\left[-\frac{1}{2x^2}\ln x\right]_1^t+\frac{1}{2}\int_1^t\frac{1}{x^3}\,dx\right)=\lim_{t\to\infty}\left(-\frac{1}{2}\frac{\ln t}{t^2}+0-\frac{1}{4t^2}+\frac{1}{4}\right)=\frac{1}{4}$$

since $\displaystyle\lim_{t\to\infty}\frac{\ln t}{t^2}\overset{\text{H}}{=}\lim_{t\to\infty}\frac{1/t}{2t}=\lim_{t\to\infty}\frac{1}{2t^2}=0.$ Convergent

23. $\displaystyle\int_{e}^{\infty}\frac{1}{x(\ln x)^3}\,dx=\lim_{t\to\infty}\int_{e}^{t}\frac{1}{x(\ln x)^3}\,dx=\lim_{t\to\infty}\int_{1}^{\ln t}u^{-3}\,du\;\left[\begin{array}{c}u=\ln x,\\ du=dx/x\end{array}\right]=\lim_{t\to\infty}\left[-\frac{1}{2u^2}\right]_1^{\ln t}$

$$=\lim_{t\to\infty}\left[-\frac{1}{2(\ln t)^2}+\frac{1}{2}\right]=0+\frac{1}{2}=\frac{1}{2}.$$ Convergent

24. $\displaystyle\int_{0}^{\infty}\frac{e^x}{e^{2x}+3}\,dx=\lim_{t\to\infty}\int_{0}^{t}\frac{e^x}{(e^x)^2+(\sqrt{3})^2}\,dx=\lim_{t\to\infty}\left[\frac{1}{\sqrt{3}}\arctan\frac{e^x}{\sqrt{3}}\right]_0^t=\frac{1}{\sqrt{3}}\lim_{t\to\infty}\left(\arctan\frac{e^t}{\sqrt{3}}-\arctan\frac{1}{\sqrt{3}}\right)$

$$=\frac{1}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}}\left(\frac{\pi}{3}\right)=\frac{\pi\sqrt{3}}{9}.$$ Convergent

25. $\displaystyle\int_{0}^{1}\frac{3}{x^5}\,dx=\lim_{t\to0^+}\int_{t}^{1}3x^{-5}\,dx=\lim_{t\to0^+}\left[-\frac{3}{4x^4}\right]_t^1=-\frac{3}{4}\lim_{t\to0^+}\left(1-\frac{1}{t^4}\right)=\infty.$ Divergent

26. $\displaystyle\int_{2}^{3}\frac{1}{\sqrt{3-x}}\,dx=\lim_{t\to3^-}\int_{2}^{t}(3-x)^{-1/2}\,dx=\lim_{t\to3^-}\left[-2(3-x)^{1/2}\right]_2^t=-2\lim_{t\to3^-}\left(\sqrt{3-t}-\sqrt{1}\right)=-2(0-1)=2.$

Convergent

27. $\displaystyle\int_{-2}^{14}\frac{dx}{\sqrt[4]{x+2}}=\lim_{t\to-2^+}\int_{t}^{14}(x+2)^{-1/4}\,dx=\lim_{t\to-2^+}\left[\frac{4}{3}(x+2)^{3/4}\right]_t^{14}=\frac{4}{3}\lim_{t\to-2^+}\left[16^{3/4}-(t+2)^{3/4}\right]$

$$=\tfrac{4}{3}(8-0)=\tfrac{32}{3}.$$ Convergent

28. $\displaystyle\int_{6}^{8}\frac{4}{(x-6)^3}\,dx=\lim_{t\to6^+}\int_{t}^{8}4(x-6)^{-3}\,dx=\lim_{t\to6^+}\left[-2(x-6)^{-2}\right]_t^8=-2\lim_{t\to6^+}\left[\frac{1}{2^2}-\frac{1}{(t-6)^2}\right]=\infty.$ Divergent

29. There is an infinite discontinuity at $x=1$. $\int_0^{33}(x-1)^{-1/5}\,dx=\int_0^1(x-1)^{-1/5}\,dx+\int_1^{33}(x-1)^{-1/5}\,dx$. Here

$\int_0^1(x-1)^{-1/5}\,dx=\lim_{t\to1^-}\int_0^t(x-1)^{-1/5}\,dx=\lim_{t\to1^-}\left[\frac{5}{4}(x-1)^{4/5}\right]_0^t=\lim_{t\to1^-}\left[\frac{5}{4}(t-1)^{4/5}-\frac{5}{4}\right]=-\frac{5}{4}$ and

$\int_1^{33}(x-1)^{-1/5}\,dx=\lim_{t\to1^+}\int_t^{33}(x-1)^{-1/5}\,dx=\lim_{t\to1^+}\left[\frac{5}{4}(x-1)^{4/5}\right]_t^{33}=\lim_{t\to1^+}\left[\frac{5}{4}\cdot16-\frac{5}{4}(t-1)^{4/5}\right]=20.$

Thus, $\int_0^{33}(x-1)^{-1/5}\,dx=-\frac{5}{4}+20=\frac{75}{4}.$ Convergent

30. $f(y) = 1/(4y-1)$ has an infinite discontinuity at $y = \frac{1}{4}$.

$$\int_{1/4}^{1} \frac{1}{4y-1}\,dy = \lim_{t\to(1/4)^+} \int_t^1 \frac{1}{4y-1}\,dy = \lim_{t\to(1/4)^+} \left[\tfrac{1}{4}\ln|4y-1|\right]_t^1 = \lim_{t\to(1/4)^+} \left[\tfrac{1}{4}\ln 3 - \tfrac{1}{4}\ln(4t-1)\right] = \infty,$$

so $\int_{1/4}^{1} \frac{1}{4y-1}\,dy$ diverges, and hence, $\int_0^1 \frac{1}{4y-1}\,dy$ diverges. Divergent

31. There is an infinite discontinuity at $x = 0$. $\int_{-1}^{1} \frac{e^x}{e^x-1}\,dx = \int_{-1}^{0} \frac{e^x}{e^x-1}\,dx + \int_0^1 \frac{e^x}{e^x-1}\,dx.$

$$\int_{-1}^{0} \frac{e^x}{e^x-1}\,dx = \lim_{t\to0^-} \int_{-1}^{t} \frac{e^x}{e^x-1}\,dx = \lim_{t\to0^-} \Big[\ln|e^x-1|\Big]_{-1}^{t} = \lim_{t\to0^-} \Big[\ln|e^t-1| - \ln|e^{-1}-1|\Big] = -\infty,$$

so $\int_{-1}^{1} \frac{e^x}{e^x-1}\,dx$ is divergent. The integral $\int_0^1 \frac{e^x}{e^x-1}\,dx$ also diverges since

$$\int_0^1 \frac{e^x}{e^x-1}\,dx = \lim_{t\to0^+} \int_t^1 \frac{e^x}{e^x-1}\,dx = \lim_{t\to0^+} \Big[\ln|e^x-1|\Big]_t^1 = \lim_{t\to0^+} \Big[\ln|e-1| - \ln|e^t-1|\Big] = \infty. \qquad \text{Divergent}$$

32. $\int_{\pi/2}^{\pi} \csc x\,dx = \lim_{t\to\pi^-} \int_{\pi/2}^{t} \csc x\,dx = \lim_{t\to\pi^-} [\ln|\csc x - \cot x|]_{\pi/2}^{t} = \lim_{t\to\pi^-} \Big[\ln(\csc t - \cot t) - \ln(1-0)\Big]$

$$= \lim_{t\to\pi^-} \ln\left(\frac{1-\cos t}{\sin t}\right) = \infty. \qquad \text{Divergent}$$

33. $I = \int_0^2 z^2 \ln z\,dz = \lim_{t\to0^+} \int_t^2 z^2 \ln z\,dz = \lim_{t\to0^+} \left[\frac{z^3}{3^2}(3\ln z - 1)\right]_t^2$ [integrate by parts or use Formula 101]

$$= \lim_{t\to0^+} \left[\tfrac{8}{9}(3\ln 2 - 1) - \tfrac{1}{9}t^3(3\ln t - 1)\right] = \tfrac{8}{3}\ln 2 - \tfrac{8}{9} - \tfrac{1}{9}\lim_{t\to0^+}\left[t^3(3\ln t - 1)\right] = \tfrac{8}{3}\ln 2 - \tfrac{8}{9} - \tfrac{1}{9}L.$$

Now $L = \lim_{t\to0^+} \left[t^3(3\ln t - 1)\right] = \lim_{t\to0^+} \frac{3\ln t - 1}{t^{-3}} \overset{\text{H}}{=} \lim_{t\to0^+} \frac{3/t}{-3/t^4} = \lim_{t\to0^+} (-t^3) = 0.$

Thus, $L = 0$ and $I = \frac{8}{3}\ln 2 - \frac{8}{9}$. Convergent

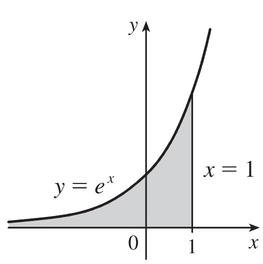
34. Integrate by parts with $u = \ln x$, $dv = dx/\sqrt{x} \quad\Rightarrow\quad du = dx/x$, $v = 2\sqrt{x}$.

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to0^+} \int_t^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to0^+} \left(\Big[2\sqrt{x}\ln x\Big]_t^1 - 2\int_t^1 \frac{dx}{\sqrt{x}}\right) = \lim_{t\to0^+} \left(-2\sqrt{t}\,\ln t - 4\Big[\sqrt{x}\Big]_t^1\right)$$

$$= \lim_{t\to0^+} \left(-2\sqrt{t}\,\ln t - 4 + 4\sqrt{t}\right) = -4$$

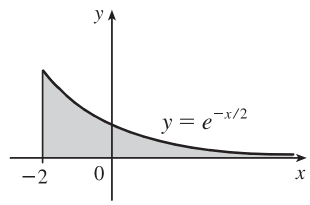
since $\lim_{t\to0^+} \sqrt{t}\,\ln t = \lim_{t\to0^+} \frac{\ln t}{t^{-1/2}} \overset{\text{H}}{=} \lim_{t\to0^+} \frac{1/t}{-t^{-3/2}/2} = \lim_{t\to0^+} \left(-2\sqrt{t}\right) = 0.$ Convergent

35.



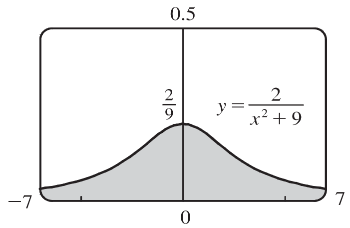
Area $= \int_{-\infty}^{1} e^x\,dx = \lim_{t\to-\infty} \left[e^x\right]_t^1 = e - \lim_{t\to-\infty} e^t = e$

36.



Area $= \int_{-2}^{\infty} e^{-x/2}\,dx = -2 \lim_{t\to\infty} \left[e^{-x/2}\right]_{-2}^{t} = -2 \lim_{t\to\infty} e^{-t/2} + 2e = 2e$

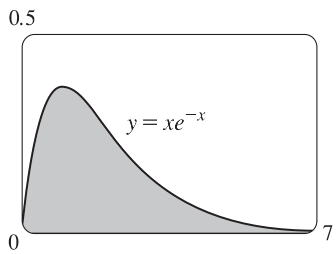
37.



$$\text{Area} = \int_{-\infty}^{\infty} \frac{2}{x^2+9}\,dx = 2\cdot 2\int_0^{\infty} \frac{1}{x^2+9}\,dx = 4\lim_{t\to\infty}\int_0^t \frac{1}{x^2+9}\,dx$$

$$= 4\lim_{t\to\infty}\left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^t = \frac{4}{3}\lim_{t\to\infty}\left[\tan^{-1}\frac{t}{3} - 0\right] = \frac{4}{3}\cdot\frac{\pi}{2} = \frac{2\pi}{3}$$

38.



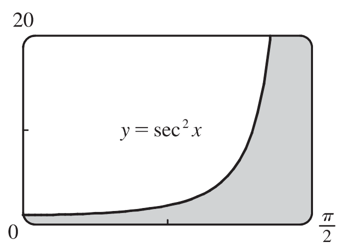
$$\text{Area} = \int_0^{\infty} xe^{-x}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x}\,dx$$

$= \lim_{t\to\infty}\left[-xe^{-x} - e^{-x}\right]_0^t$ [use parts with $u = x$ and $dv = e^{-x}\,dx$]

$= \lim_{t\to\infty}\left[(-te^{-t} - e^{-t}) - (-1)\right]$

$= 0$ [use l'Hospital's Rule] $-0 + 1 = 1$

39.

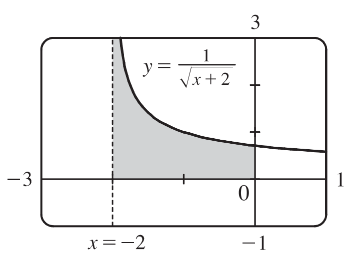


$$\text{Area} = \int_0^{\pi/2} \sec^2 x\,dx = \lim_{t\to(\pi/2)^-}\int_0^t \sec^2 x\,dx = \lim_{t\to(\pi/2)^-}\left[\tan x\right]_0^t$$

$$= \lim_{t\to(\pi/2)^-}(\tan t - 0) = \infty$$

Infinite area

40.



$$\text{Area} = \int_{-2}^{0}\frac{1}{\sqrt{x+2}}\,dx = \lim_{t\to-2^+}\int_t^0 \frac{1}{\sqrt{x+2}}\,dx = \lim_{t\to-2^+}\left[2\sqrt{x+2}\right]_t^0$$

$$= \lim_{t\to-2^+}\left(2\sqrt{2} - 2\sqrt{t+2}\right) = 2\sqrt{2} - 0 = 2\sqrt{2}$$

41. (a)

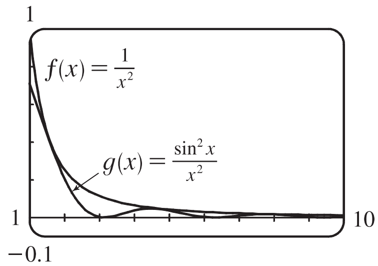
t	$\int_1^t g(x)\,dx$
2	0.447453
5	0.577101
10	0.621306
100	0.668479
1000	0.672957
10,000	0.673407

$g(x) = \dfrac{\sin^2 x}{x^2}$.

It appears that the integral is convergent.

(b) $-1 \le \sin x \le 1 \;\Rightarrow\; 0 \le \sin^2 x \le 1 \;\Rightarrow\; 0 \le \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$. Since $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent [Equation 2 with $p = 2 > 1$], $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent by the Comparison Theorem.

(c)



Since $\int_1^\infty f(x)\,dx$ is finite and the area under $g(x)$ is less than the area under $f(x)$ on any interval $[1, t]$, $\int_1^\infty g(x)\,dx$ must be finite; that is, the integral is convergent.

42. (a)

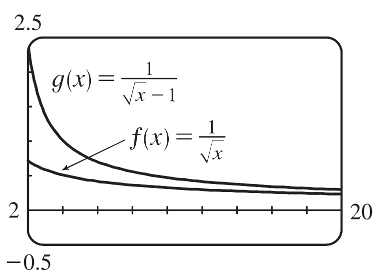
t	$\int_2^t g(x)dx$
5	3.830327
10	6.801200
100	23.328769
1000	69.023361
10,000	208.124560

$g(x) = \dfrac{1}{\sqrt{x} - 1}$.

It appears that the integral is divergent.

(b) For $x \ge 2$, $\sqrt{x} > \sqrt{x} - 1 \;\Rightarrow\; \dfrac{1}{\sqrt{x}} < \dfrac{1}{\sqrt{x} - 1}$. Since $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}}\,dx$ is divergent [Equation 2 with $p = \frac{1}{2} \le 1$], $\displaystyle\int_2^\infty \frac{1}{\sqrt{x} - 1}\,dx$ is divergent by the Comparison Theorem.

(c)



Since $\int_2^\infty f(x)\,dx$ is infinite and the area under $g(x)$ is greater than the area under $f(x)$ on any interval $[2, t]$, $\int_2^\infty g(x)\,dx$ must be infinite; that is, the integral is divergent.

43. For $x > 0$, $\dfrac{x}{x^3 + 1} < \dfrac{x}{x^3} = \dfrac{1}{x^2}$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by Equation 2 with $p = 2 > 1$, so $\displaystyle\int_1^\infty \frac{x}{x^3 + 1}\,dx$ is convergent by the Comparison Theorem. $\displaystyle\int_0^1 \frac{x}{x^3 + 1}\,dx$ is a constant, so $\displaystyle\int_0^\infty \frac{x}{x^3 + 1}\,dx = \int_0^1 \frac{x}{x^3 + 1}\,dx + \int_1^\infty \frac{x}{x^3 + 1}\,dx$ is also convergent.

44. For $x \ge 1$, $\dfrac{2 + e^{-x}}{x} > \dfrac{2}{x}$ [since $e^{-x} > 0$] $> \dfrac{1}{x}$. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ is divergent by Equation 2 with $p = 1 \le 1$, so $\displaystyle\int_1^\infty \frac{2 + e^{-x}}{x}\,dx$ is divergent by the Comparison Theorem.

45. For $x > 1$, $f(x) = \dfrac{x + 1}{\sqrt{x^4 - x}} > \dfrac{x + 1}{\sqrt{x^4}} > \dfrac{x}{x^2} = \dfrac{1}{x}$, so $\displaystyle\int_2^\infty f(x)\,dx$ diverges by comparison with $\displaystyle\int_2^\infty \frac{1}{x}\,dx$, which diverges by Equation 2 with $p = 1 \le 1$. Thus, $\int_1^\infty f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^\infty f(x)\,dx$ also diverges.

46. For $x \geq 0$, $\arctan x < \frac{\pi}{2} < 2$, so $\dfrac{\arctan x}{2+e^x} < \dfrac{2}{2+e^x} < \dfrac{2}{e^x} = 2e^{-x}$. Now

$I = \displaystyle\int_0^\infty 2e^{-x}\,dx = \lim_{t\to\infty}\int_0^t 2e^{-x}\,dx = \lim_{t\to\infty}\left[-2e^{-x}\right]_0^t = \lim_{t\to\infty}\left(-\frac{2}{e^t}+2\right) = 2$, so I is convergent, and by comparison,

$\displaystyle\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$ is convergent.

47. For $0 < x \leq 1$, $\dfrac{\sec^2 x}{x\sqrt{x}} > \dfrac{1}{x^{3/2}}$. Now

$I = \displaystyle\int_0^1 x^{-3/2}\,dx = \lim_{t\to 0^+}\int_t^1 x^{-3/2}\,dx = \lim_{t\to 0^+}\left[-2x^{-1/2}\right]_t^1 = \lim_{t\to 0^+}\left(-2+\frac{2}{\sqrt{t}}\right) = \infty$, so I is divergent, and by

comparison, $\displaystyle\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}$ is divergent.

48. For $0 < x \leq 1$, $\dfrac{\sin^2 x}{\sqrt{x}} \leq \dfrac{1}{\sqrt{x}}$. Now

$I = \displaystyle\int_0^\pi \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^\pi x^{-1/2}\,dx = \lim_{t\to 0^+}\left[2x^{1/2}\right]_t^\pi = \lim_{t\to 0^+}\left(2\pi - 2\sqrt{t}\right) = 2\pi - 0 = 2\pi$, so I is convergent, and by

comparison, $\displaystyle\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx$ is convergent.

49. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \int_0^1 \frac{dx}{\sqrt{x}\,(1+x)} + \int_1^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{\sqrt{x}\,(1+x)} + \lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt{x}\,(1+x)}$. Now

$\displaystyle\int \frac{dx}{\sqrt{x}\,(1+x)} = \int \frac{2u\,du}{u(1+u^2)}$ $\left[\begin{matrix} u=\sqrt{x},\, x=u^2, \\ dx = 2u\,du \end{matrix}\right]$ $\displaystyle = 2\int \frac{du}{1+u^2} = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{x} + C$, so

$$\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to 0^+}\left[2\tan^{-1}\sqrt{x}\right]_t^1 + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{x}\right]_1^t$$
$$= \lim_{t\to 0^+}\left[2\left(\tfrac{\pi}{4}\right) - 2\tan^{-1}\sqrt{t}\right] + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{t} - 2\left(\tfrac{\pi}{4}\right)\right] = \tfrac{\pi}{2} - 0 + 2\left(\tfrac{\pi}{2}\right) - \tfrac{\pi}{2} = \pi.$$

50. Let $u = \ln x$. Then $du = dx/x$ $\Rightarrow$ $\displaystyle\int_e^\infty \frac{dx}{x\,(\ln x)^p} = \int_1^\infty \frac{du}{u^p}$. By Example 4, this converges to $\dfrac{1}{p-1}$ if $p > 1$

and diverges otherwise.

51. If $p = 1$, then $\displaystyle\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{x} = \lim_{t\to 0^+}\left[\ln x\right]_t^1 = \infty.$ Divergent.

If $p \neq 1$, then $\displaystyle\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{x^p}$ [note that the integral is not improper if $p < 0$]

$$= \lim_{t\to 0^+}\left[\frac{x^{-p+1}}{-p+1}\right]_t^1 = \lim_{t\to 0^+}\frac{1}{1-p}\left[1 - \frac{1}{t^{p-1}}\right]$$

If $p > 1$, then $p - 1 > 0$, so $\dfrac{1}{t^{p-1}} \to \infty$ as $t \to 0^+$, and the integral diverges.

If $p < 1$, then $p - 1 < 0$, so $\dfrac{1}{t^{p-1}} \to 0$ as $t \to 0^+$ and $\displaystyle\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p}\left[\lim_{t\to 0^+}\left(1 - t^{1-p}\right)\right] = \frac{1}{1-p}$.

Thus, the integral converges if and only if $p < 1$, and in that case its value is $\dfrac{1}{1-p}$.

52. (a) $\boldsymbol{n=0}$: $\displaystyle\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t = \lim_{t\to\infty}\left[-e^{-t}+1\right] = 0+1 = 1$

$\boldsymbol{n=1}$: $\displaystyle\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x}\,dx$. To evaluate $\displaystyle\int xe^{-x}\,dx$, we'll use integration by parts with $u=x$, $dv=e^{-x}\,dx \;\Rightarrow\; du=dx$, $v=-e^{-x}$.

So $\displaystyle\int xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx = -xe^{-x} - e^{-x} + C = (-x-1)e^{-x} + C$ and

$$\lim_{t\to\infty}\int_0^t xe^{-x}\,dx = \lim_{t\to\infty}\left[(-x-1)e^{-x}\right]_0^t = \lim_{t\to\infty}\left[(-t-1)e^{-t}+1\right] = \lim_{t\to\infty}\left[-te^{-t}-e^{-t}+1\right]$$

$$= 0-0+1 \quad [\text{use l'Hospital's Rule}] \quad = 1$$

$\boldsymbol{n=2}$: $\displaystyle\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx$. To evaluate $\displaystyle\int x^2e^{-x}\,dx$, we could use integration by parts again or Formula 97. Thus,

$$\lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx = \lim_{t\to\infty}\left[-x^2e^{-x}\right]_0^t + 2\lim_{t\to\infty}\int_0^t xe^{-x}\,dx$$

$$= 0+0+2(1) \quad [\text{use l'Hospital's Rule and the result for } n=1] \quad = 2$$

$\boldsymbol{n=3}$: $\displaystyle\int_0^\infty x^n e^{-x}dx = \lim_{t\to\infty}\int_0^t x^3e^{-x}\,dx \overset{97}{=} \lim_{t\to\infty}\left[-x^3e^{-x}\right]_0^t + 3\lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx$

$$= 0+0+3(2) \quad [\text{use l'Hospital's Rule and the result for } n=2] \quad = 6$$

(b) For $n=1$, 2, and 3, we have $\int_0^\infty x^ne^{-x}\,dx = 1$, 2, and 6. The values for the integral are equal to the factorials for n, so we guess $\int_0^\infty x^ne^{-x}\,dx = n!$.

(c) Suppose that $\int_0^\infty x^ke^{-x}\,dx = k!$ for some positive integer k. Then $\int_0^\infty x^{k+1}e^{-x}\,dx = \lim\limits_{t\to\infty}\int_0^t x^{k+1}e^{-x}\,dx$.
To evaluate $\int x^{k+1}e^{-x}\,dx$, we use parts with $u=x^{k+1}$, $dv=e^{-x}\,dx \;\Rightarrow\; du=(k+1)x^k\,dx$, $v=-e^{-x}$.
So $\int x^{k+1}e^{-x}dx = -x^{k+1}e^{-x} - \int -(k+1)x^ke^{-x}\,dx = -x^{k+1}e^{-x} + (k+1)\int x^ke^{-x}\,dx$ and

$$\lim_{t\to\infty}\int_0^t x^{k+1}e^{-x}\,dx = \lim_{t\to\infty}\left[-x^{k+1}e^{-x}\right]_0^t + (k+1)\lim_{t\to\infty}\int_0^t x^ke^{-x}\,dx$$

$$= \lim_{t\to\infty}\left[-t^{k+1}e^{-t}+0\right] + (k+1)k! = 0+0+(k+1)! = (k+1)!,$$

so the formula holds for $k+1$. By induction, the formula holds for all positive integers. (Since $0!=1$, the formula holds for $n=0$, too.)

53. (a) $I = \int_{-\infty}^\infty x\,dx = \int_{-\infty}^0 x\,dx + \int_0^\infty x\,dx$, and $\int_0^\infty x\,dx = \lim\limits_{t\to\infty}\int_0^t x\,dx = \lim\limits_{t\to\infty}\left[\frac12x^2\right]_0^t = \lim\limits_{t\to\infty}\left[\frac12t^2-0\right] = \infty$, so I is divergent.

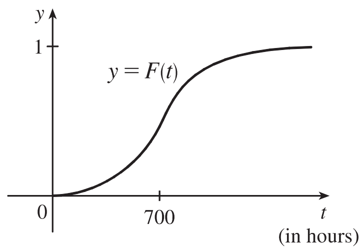
(b) $\int_{-t}^t x\,dx = \left[\frac12x^2\right]_{-t}^t = \frac12t^2 - \frac12t^2 = 0$, so $\lim\limits_{t\to\infty}\int_{-t}^t x\,dx = 0$. Therefore, $\int_{-\infty}^\infty x\,dx \neq \lim\limits_{t\to\infty}\int_{-t}^t x\,dx$.

54. Assume without loss of generality that $a<b$. Then

$$\begin{aligned}\int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx &= \lim_{t\to-\infty}\int_t^a f(x)\,dx + \lim_{u\to\infty}\int_a^u f(x)\,dx\\
&= \lim_{t\to-\infty}\int_t^a f(x)\,dx + \lim_{u\to\infty}\left[\int_a^b f(x)\,dx + \int_b^u f(x)\,dx\right]\\
&= \lim_{t\to-\infty}\int_t^a f(x)\,dx + \int_a^b f(x)\,dx + \lim_{u\to\infty}\int_b^u f(x)\,dx\\
&= \lim_{t\to-\infty}\left[\int_t^a f(x)\,dx + \int_a^b f(x)\,dx\right] + \int_b^\infty f(x)\,dx\\
&= \lim_{t\to-\infty}\int_t^b f(x)\,dx + \int_b^\infty f(x)\,dx = \int_{-\infty}^b f(x)\,dx + \int_b^\infty f(x)\,dx\end{aligned}$$

55. We would expect a small percentage of bulbs to burn out in the first few hundred hours, most of the bulbs to burn out after close to 700 hours, and a few overachievers to burn on and on.

(a)



(b) $r(t) = F'(t)$ is the rate at which the fraction $F(t)$ of burnt-out bulbs increases as t increases. This could be interpreted as a fractional burnout rate.

(c) $\int_0^\infty r(t)\,dt = \lim\limits_{x\to\infty} F(x) = 1$, since all of the bulbs will eventually burn out.

56. Let $k = \dfrac{M}{2RT}$ so that $\overline{v} = \dfrac{4}{\sqrt{\pi}}\,k^{3/2}\displaystyle\int_0^\infty v^3 e^{-kv^2}\,dv$. Let I denote the integral and use parts to integrate I. Let $\alpha = v^2$, $d\beta = ve^{-kv^2}\,dv \quad\Rightarrow\quad d\alpha = 2v\,dv$, $\beta = -\dfrac{1}{2k}e^{-kv^2}$:

$$I = \lim_{t\to\infty}\left[-\frac{1}{2k}v^2e^{-kv^2}\right]_0^t + \frac{1}{k}\int_0^\infty ve^{-kv^2}\,dv_0^t = -\frac{1}{2k}\lim_{t\to\infty}\left(t^2e^{-kt^2}\right) + \frac{1}{k}\lim_{t\to\infty}\left[-\frac{1}{2k}e^{-kv^2}\right]$$

$$\overset{\text{H}}{=} -\frac{1}{2k}\cdot 0 - \frac{1}{2k^2}(0-1) = \frac{1}{2k^2}$$

Thus, $\overline{v} = \dfrac{4}{\sqrt{\pi}}k^{3/2}\cdot\dfrac{1}{2k^2} = \dfrac{2}{(k\pi)^{1/2}} = \dfrac{2}{[\pi M/(2RT)]^{1/2}} = \dfrac{2\sqrt{2}\sqrt{RT}}{\sqrt{\pi M}} = \sqrt{\dfrac{8RT}{\pi M}}$.

57. $I = \displaystyle\int_0^\infty te^{kt}\,dt = \lim_{s\to\infty}\left[\frac{1}{k^2}(kt-1)e^{kt}\right]_0^s$ [Formula 96, or parts] $= \displaystyle\lim_{s\to\infty}\left[\left(\frac{1}{k}se^{ks} - \frac{1}{k^2}e^{ks}\right) - \left(-\frac{1}{k^2}\right)\right]$.

Since $k < 0$ the first two terms approach 0 (you can verify that the first term does so with l'Hospital's Rule), so the limit is equal to $1/k^2$. Thus, $M = -kI = -k(1/k^2) = -1/k = -1/(-0.000121) \approx 8264.5$ years.

58. $y(s) = \displaystyle\int_s^R \frac{2r}{\sqrt{r^2-s^2}}\,x(r)\,dr$ and $x(r) = \frac{1}{2}(R-r)^2 \quad\Rightarrow$

$$y(s) = \lim_{t\to s^+}\int_t^R \frac{r(R-r)^2}{\sqrt{r^2-s^2}}\,dr = \lim_{t\to s^+}\int_t^R \frac{r^3-2Rr^2+R^2r}{\sqrt{r^2-s^2}}\,dr$$

$$= \lim_{t\to s^+}\left[\int_t^R \frac{r^3\,dr}{\sqrt{r^2-s^2}} - 2R\int_t^R \frac{r^2\,dr}{\sqrt{r^2-s^2}} + R^2\int_t^R \frac{r\,dr}{\sqrt{r^2-s^2}}\right] = \lim_{t\to s^+}\left(I_1 - 2RI_2 + R^2I_3\right) = L$$

For I_1: Let $u = \sqrt{r^2-s^2} \quad\Rightarrow\quad u^2 = r^2-s^2$, $r^2 = u^2+s^2$, $2r\,dr = 2u\,du$, so, omitting limits and constant of integration,

$$I_1 = \int \frac{(u^2+s^2)u}{u}\,du = \int (u^2+s^2)\,du = \tfrac{1}{3}u^3 + s^2u = \tfrac{1}{3}u(u^2+3s^2)$$
$$= \tfrac{1}{3}\sqrt{r^2-s^2}\,(r^2-s^2+3s^2) = \tfrac{1}{3}\sqrt{r^2-s^2}\,(r^2+2s^2)$$

For I_2: Using Formula 44, $I_2 = \dfrac{r}{2}\sqrt{r^2-s^2} + \dfrac{s^2}{2}\ln\left|r+\sqrt{r^2-s^2}\right|$.

For I_3: Let $u = r^2-s^2 \quad\Rightarrow\quad du = 2r\,dr$. Then $I_3 = \dfrac{1}{2}\displaystyle\int\frac{du}{\sqrt{u}} = \frac{1}{2}\cdot 2\sqrt{u} = \sqrt{r^2-s^2}$.

Thus,

$$L = \lim_{t\to s^+}\left[\tfrac{1}{3}\sqrt{r^2-s^2}\,(r^2+2s^2) - 2R\left(\frac{r}{2}\sqrt{r^2-s^2}+\frac{s^2}{2}\ln\left|r+\sqrt{r^2-s^2}\,\right|\right)+R^2\sqrt{r^2-s^2}\,\right]_t^R$$

$$= \lim_{t\to s^+}\left[\tfrac{1}{3}\sqrt{R^2-s^2}(R^2+2s^2) - 2R\left(\frac{R}{2}\sqrt{R^2-s^2}+\frac{s^2}{2}\ln\left|R+\sqrt{R^2-s^2}\,\right|\right)+R^2\sqrt{R^2-s^2}\,\right]$$

$$- \lim_{t\to s^+}\left[\tfrac{1}{3}\sqrt{t^2-s^2}\,(t^2+2s^2) - 2R\left(\frac{t}{2}\sqrt{t^2-s^2}+\frac{s^2}{2}\ln\left|t+\sqrt{t^2-s^2}\,\right|\right)+R^2\sqrt{t^2-s^2}\,\right]$$

$$= \left[\tfrac{1}{3}\sqrt{R^2-s^2}\,(R^2+2s^2) - Rs^2\ln\left|R+\sqrt{R^2-s^2}\,\right|\right] - \left[-Rs^2\ln|s|\right]$$

$$= \tfrac{1}{3}\sqrt{R^2-s^2}\,(R^2+2s^2) - Rs^2\ln\left(\frac{R+\sqrt{R^2-s^2}}{s}\right)$$

59. $I = \displaystyle\int_a^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty}\int_a^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty}\left[\tan^{-1}x\right]_a^t = \lim_{t\to\infty}\left(\tan^{-1}t - \tan^{-1}a\right) = \frac{\pi}{2} - \tan^{-1}a.$

$I < 0.001 \quad\Rightarrow\quad \frac{\pi}{2} - \tan^{-1}a < 0.001 \quad\Rightarrow\quad \tan^{-1}a > \frac{\pi}{2} - 0.001 \quad\Rightarrow\quad a > \tan\left(\frac{\pi}{2} - 0.001\right) \approx 1000.$

60. $f(x) = e^{-x^2}$ and $\Delta x = \frac{4-0}{8} = \frac{1}{2}$.

$\int_0^4 f(x)\,dx \approx S_8 = \frac{1}{2\cdot 3}[f(0)+4f(0.5)+2f(1)+\cdots+2f(3)+4f(3.5)+f(4)] \approx \frac{1}{6}(5.31717808) \approx 0.8862$

Now $x > 4 \quad\Rightarrow\quad -x\cdot x < -x\cdot 4 \quad\Rightarrow\quad e^{-x^2} < e^{-4x} \quad\Rightarrow\quad \int_4^\infty e^{-x^2}\,dx < \int_4^\infty e^{-4x}\,dx.$

$\int_4^\infty e^{-4x}\,dx = \lim_{t\to\infty}\left[-\frac{1}{4}e^{-4x}\right]_4^t = -\frac{1}{4}\left(0 - e^{-16}\right) = 1/(4e^{16}) \approx 0.0000000281 < 0.0000001$, as desired.

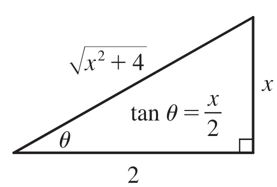
61. We use integration by parts: let $u = x$, $dv = xe^{-x^2}\,dx \quad\Rightarrow\quad du = dx$, $v = -\frac{1}{2}e^{-x^2}$. So

$$\int_0^\infty x^2e^{-x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2}xe^{-x^2}\right]_0^t + \frac{1}{2}\int_0^\infty e^{-x^2}\,dx = \lim_{t\to\infty}\left[-\frac{t}{2e^{t^2}}\right] + \frac{1}{2}\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-x^2}\,dx$$

(The limit is 0 by l'Hospital's Rule.)

62. $\int_0^\infty e^{-x^2}\,dx$ is the area under the curve $y = e^{-x^2}$ for $0 \le x < \infty$ and $0 < y \le 1$. Solving $y = e^{-x^2}$ for x, we get $y = e^{-x^2} \quad\Rightarrow\quad \ln y = -x^2 \quad\Rightarrow\quad -\ln y = x^2 \quad\Rightarrow\quad x = \pm\sqrt{-\ln y}$. Since x is positive, choose $x = \sqrt{-\ln y}$, and the area is represented by $\int_0^1 \sqrt{-\ln y}\,dy$. Therefore, each integral represents the same area, so the integrals are equal.

63. For the first part of the integral, let $x = 2\tan\theta \quad\Rightarrow\quad dx = 2\sec^2\theta\,d\theta$.



$$\int \frac{1}{\sqrt{x^2+4}}\,dx = \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|.$$

From the figure, $\tan\theta = \dfrac{x}{2}$, and $\sec\theta = \dfrac{\sqrt{x^2+4}}{2}$. So

$$I = \int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2}\right)dx = \lim_{t\to\infty}\left[\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| - C\ln|x+2|\right]_0^t$$

$$= \lim_{t\to\infty}\left[\ln\frac{\sqrt{t^2+4}+t}{2} - C\ln(t+2) - (\ln 1 - C\ln 2)\right]$$

$$= \lim_{t\to\infty}\left[\ln\left(\frac{\sqrt{t^2+4}+t}{2\,(t+2)^C}\right) + \ln 2^C\right] = \ln\left(\lim_{t\to\infty}\frac{t+\sqrt{t^2+4}}{(t+2)^C}\right) + \ln 2^{C-1}$$

[continued]

Now $L = \lim\limits_{t\to\infty} \dfrac{t+\sqrt{t^2+4}}{(t+2)^C} \overset{\text{H}}{=} \lim\limits_{t\to\infty} \dfrac{1+t/\sqrt{t^2+4}}{C\,(t+2)^{C-1}} = \dfrac{2}{C \lim\limits_{t\to\infty} (t+2)^{C-1}}$.

If $C < 1$, $L = \infty$ and I diverges.

If $C = 1$, $L = 2$ and I converges to $\ln 2 + \ln 2^0 = \ln 2$.

If $C > 1$, $L = 0$ and I diverges to $-\infty$.

64. $I = \displaystyle\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1}\right) dx = \lim_{t\to\infty} \left[\tfrac{1}{2}\ln(x^2+1) - \tfrac{1}{3}C\ln(3x+1)\right]_0^t = \lim_{t\to\infty} \left[\ln(t^2+1)^{1/2} - \ln(3t+1)^{C/3}\right]$

$$= \lim_{t\to\infty}\left(\ln \frac{(t^2+1)^{1/2}}{(3t+1)^{C/3}}\right) = \ln\left(\lim_{t\to\infty} \frac{\sqrt{t^2+1}}{(3t+1)^{C/3}}\right)$$

For $C \le 0$, the integral diverges. For $C > 0$, we have

$$L = \lim_{t\to\infty} \frac{\sqrt{t^2+1}}{(3t+1)^{C/3}} \overset{\text{H}}{=} \lim_{t\to\infty} \frac{t/\sqrt{t^2+1}}{C(3t+1)^{(C/3)-1}} = \frac{1}{C}\lim_{t\to\infty} \frac{1}{(3t+1)^{(C/3)-1}}$$

For $C/3 < 1 \quad\Leftrightarrow\quad C < 3$, $L = \infty$ and I diverges.

For $C = 3$, $L = \frac{1}{3}$ and $I = \ln \frac{1}{3}$.

For $C > 3$, $L = 0$ and I diverges to $-\infty$.

65. No, $I = \int_0^\infty f(x)\,dx$ must be *divergent*. Since $\lim\limits_{x\to\infty} f(x) = 1$, there must exist an N such that if $x \ge N$, then $f(x) \ge \frac{1}{2}$.

Thus, $I = I_1 + I_2 = \int_0^N f(x)\,dx + \int_N^\infty f(x)\,dx$, where I_1 is an ordinary definite integral that has a finite value, and I_2 is improper and diverges by comparison with the divergent integral $\int_N^\infty \frac{1}{2}\,dx$.

66. As in Exercise 49, we let $I = \displaystyle\int_0^\infty \frac{x^a}{1+x^b}\,dx = I_1 + I_2$, where $I_1 = \displaystyle\int_0^1 \frac{x^a}{1+x^b}\,dx$ and $I_2 = \displaystyle\int_1^\infty \frac{x^a}{1+x^b}\,dx$. We will show that I_1 converges for $a > -1$ and I_2 converges for $b > a+1$, so that I converges when $a > -1$ and $b > a+1$.

I_1 is improper only when $a < 0$. When $0 \le x \le 1$, we have $\dfrac{1}{1+x^b} \le 1 \quad\Rightarrow\quad \dfrac{1}{x^{-a}(1+x^b)} \le \dfrac{1}{x^{-a}}$. The integral $\displaystyle\int_0^1 \frac{1}{x^{-a}}\,dx$ converges for $-a < 1$ [or $a > -1$] by Exercise 51, so by the Comparison Theorem, $\displaystyle\int_0^1 \frac{1}{x^{-a}(1+x^b)}\,dx$ converges for $-1 < a < 0$. I_1 is not improper when $a \ge 0$, so it has a finite real value in that case. Therefore, I_1 has a finite real value (converges) when $a > -1$.

I_2 is always improper. When $x \ge 1$, $\dfrac{x^a}{1+x^b} = \dfrac{1}{x^{-a}(1+x^b)} = \dfrac{1}{x^{-a}+x^{b-a}} < \dfrac{1}{x^{b-a}}$. By (2), $\displaystyle\int_1^\infty \frac{1}{x^{b-a}}\,dx$ converges for $b - a > 1$ (or $b > a+1$), so by the Comparison Theorem, $\displaystyle\int_1^\infty \frac{x^a}{1+x^b}\,dx$ converges for $b > a+1$.

Thus, I converges if $a > -1$ and $b > a+1$.