Test 2

Math 450

3.13.12

CWID: KEY

1. Suppose we draw a sample of size 20 from an EXP(θ) population, and in so doing we get a sample mean of 12. Find the 95% left and right confidence limits for θ.

$$L = 2n\bar{x} / \chi^2_{.975}(2n) = 8.09$$

$$R = 2n\bar{x} / \chi^2_{.025}(2n) = 19.65$$

2. Suppose we draw a LARGE sample of size n from a Bin (1,p) population. Derive the formulas for the left and right 1- α level confidence limits of p.

$X_1 \dots X_n$ iid $\text{Bin}(1, p)$ $\quad \mu = p$

$\sigma^2 = pq$

$\bar{X} \approx N\left(p, \frac{pq}{n}\right) \qquad \frac{\bar{X} - p}{\sqrt{\frac{pq}{n}}} \approx Z$

$$1-\alpha = P\left(-z_{1-\frac{\alpha}{2}} < \frac{\bar{X} - p}{\sqrt{\frac{pq}{n}}} < z_{1-\frac{\alpha}{2}}\right)$$

So

$$1-\alpha = P\left(\bar{X} - z_{1-\frac{\alpha}{2}}\sqrt{\frac{pq}{n}} < p < \bar{X} + z_{1-\frac{\alpha}{2}}\sqrt{\frac{pq}{n}}\right)$$

let $\hat{p} = \bar{X}$ $\quad \hat{q} = 1 - \bar{X}$ $\quad$ then

$$1-\alpha = P\left(\hat{p} - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{p}\hat{q}}{n}} < p < \hat{p} + z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{p}\hat{q}}{n}}\right)$$

$\uparrow$ L $\qquad\qquad \uparrow$ R

3. Given two populations: $N(\mu_1,\sigma^2)$ and $N(\mu_2,\sigma^2)$, from which we draw random samples of size 8 and 5, respectively, find the two sided 99% confidence limits for $\mu_2 - \mu_1$ given the following information: $\bar{X}_1 = 100$, $S_1^2 = 5$, $\bar{X}_2 = 80$, $S_2^2 = 4.8$.

$$L, R = \bar{X}_2 - \bar{X}_1 \mp t_{.995}(n_1+n_2-2)\, S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

where $$S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}$$

So

$$L, R = -20 \mp t_{.995}(11) \sqrt{4.92} \sqrt{.325}$$

$$= -20 \mp 3.106 \cdot 1.599$$

$$= -20 \mp 4.966$$

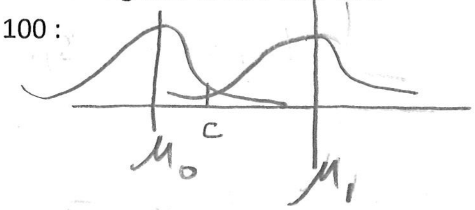
$$= (-24.966, -15.034)$$

4.

a. Derive the formula for the sample size that will assure type I and II errors of α and β, respectively, in a test of hypothesis of μ_0 vs. μ_1 (with $\mu_0 < \mu_1$) in a Normal population with known variance σ^2.

b. Provide an ACCURATE sketch of the power curve for the following test of size 0.05 in a population which is N(μ, 25) and the sample size is n = 100 :

i. H_0: μ = 10 vs. H_a: μ > 10



a) $\alpha = P(\bar{X} > c \mid \mu = \mu_0)$ $\qquad \dfrac{c - \mu_0}{\frac{\sigma}{\sqrt{n}}} = z_{1-\alpha}$

$\beta = P(\bar{X} < c \mid \mu = \mu_1)$ $\qquad \dfrac{c - \mu_1}{\frac{\sigma}{\sqrt{n}}} = -z_{1-\beta}$

So $c = \mu_0 + z_{1-\alpha}\dfrac{\sigma}{\sqrt{n}} = \mu_1 - z_{1-\beta}\dfrac{\sigma}{\sqrt{n}}$

$\therefore \quad \dfrac{\sigma}{\sqrt{n}}(z_{1-\alpha} + z_{1-\beta}) = \mu_1 - \mu_0$

$$\sqrt{n} = \frac{\sigma(z_{1-\alpha} + z_{1-\beta})}{\mu_1 - \mu_0}$$

$$n = \frac{\sigma^2 (z_{1-\alpha} + z_{1-\beta})^2}{(\mu_1 - \mu_0)^2}$$

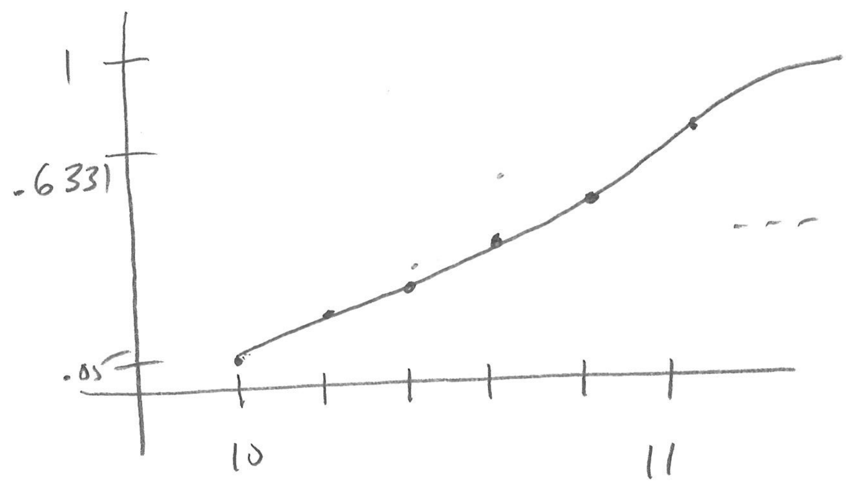
4 b) $$\pi(\mu) = P\left(\frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} > z_{1-\alpha} \,\middle|\, \mu \right)$$

$$= P\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} > z_{1-\alpha} + \frac{\mu_0 - \mu}{\frac{\sigma}{\sqrt{n}}} \,\middle|\, \mu \right)$$

so $$\pi(\mu) = 1 - \Phi\left(z_{.95} + \frac{10 - \mu}{.5} \right)$$

$$= 1 - \Phi\left(1.645 + 2(10-\mu)\right)$$

μ	$\pi(\mu)$
10	.0500
10.2	.1075
10.4	.2005
10.6	.3300
10.8	.4840
11.0	.6331
⋮	⋮



5. Suppose we wish to test whether $\sigma^2 = 25$ vs. σ^2 <> 25 in a normal population based on the following data: $\bar{X} = 35$, $S^2 = 11.84$, $n = 20$.

   a. Find the p-value.

   b. Will we reject the null hypothesis at the 3% level of significance?

a) $P\left(\frac{(n-1)S^2}{\sigma_0^2} < \frac{19 \cdot 11.84}{25} \right)$

$= P\left(\chi^2(19) < 8.9984 \right)$

≈ 0.025

p-value ≈ 0.05

b) No, accept H_0 at 3% level

6. Nike wants to compare two brands of shoes (A and B) to see if they have significantly different durability. Accordingly, they select 16 runners for their experiment. Each runner flips a coin to determine which brand to wear on each foot. Then each runner uses these two shoes for 3 months of training. After this period, Nike measures how much wear each shoe has generated, forming the following test statistics: $\bar{D} = \bar{A} - \bar{B} = 5$, $S_d = 12$. Based on these data, should Nike accept the claim that the shoes are identically durable? Use $\alpha = 0.05$.

$H_0 : \mu_d = 0$

Rej H_0 if $\left| \dfrac{\bar{D} - 0}{\frac{S_d}{\sqrt{n}}} \right| > t_{1-\frac{\alpha}{2}}(n-1)$

$\dfrac{\bar{D} - 0}{\frac{S_d}{\sqrt{n}}} = \dfrac{5}{3} = 1.66$; $t_{.975}(15) = 2.131$

Accept H_0

7. Let $x_1, \ldots, x_{10}$ be an observed random sample from a POI(μ) population. Use the chi-square distribution technique to test the following: H_0: $\mu = 1$, vs. H_a: $\mu > 1$ when $\Sigma x = 15$. Use $\alpha = 0.05$.

Reject H_0 if $2n\mu_0 \le \chi^2_{1-\alpha}(2s)$

or if $20 \le \chi^2_{.05}(30) = 18.49$

No— So Accept H_0.

8. Find the **exact** form of the most powerful test of size α for testing $\sigma = \sigma_0$ vs. $\sigma = \sigma_1$ ($\sigma_0 > \sigma_1$) in a $N(0, \sigma^2)$ population.

$$\lambda = \frac{\left(\frac{1}{\sqrt{2\pi}\,\sigma_0}\right)^n e^{-\Sigma x_i^2/2\sigma_0^2}}{\left(\frac{1}{\sqrt{2\pi}\,\sigma_1}\right)^n e^{-\Sigma x_i^2/2\sigma_1^2}} \le k$$

or $$\left(\frac{\sigma_1}{\sigma_0}\right)^n e^{-\Sigma x_i^2\left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}\right)} \le k$$

Take ln & clean up → (use $\sigma_0 > \sigma_1$)

$$\Sigma x_i^2 \ge K \quad , K = ?$$

We need $P(\Sigma X_i^2 \ge K) = \alpha$

or $P(\Sigma X_i^2/\sigma_0^2 \ge k) = \alpha$

or $P(\chi^2(n) \ge k) = \alpha$

$K = \chi^2_{1-\alpha}(n)$ so

Reject H_0 if $\Sigma X_i^2/\sigma_0^2 \ge \chi^2_{1-\alpha}(n)$