THEORETICAL EXERCISES

1. Decide in each case whether the given hypothesis is simple or composite:
   (a) the hypothesis that a random variable has the gamma distribution (4.3.6) with $\alpha = 3$ and $\beta = 2$;
   (b) the hypothesis that a random variable has the gamma distribution (4.3.6) with $\alpha = 3$ and $\beta \neq 2$;
   (c) the hypothesis that a random variable has an exponential density;
   (d) the hypothesis that a random variable has the beta distribution with the mean $\mu = 0.50$;
   (e) the hypothesis that a random variable has a Poisson distribution with $\lambda = 1.25$;
   (f) the hypothesis that a random variable has a Poisson distribution with $\lambda > 1.25$;
   (g) the hypothesis that a random variable has a normal distribution with the mean $\mu = 100$;
   (h) the hypothesis that a random variable has the negative binomial distribution with $k = 3$ and $\theta < 0.60$.
2. A bowl contains seven marbles of which θ are red while the others are blue. In order to test the null hypothesis $\theta = 2$ against the alternative $\theta = 4$, two of the marbles are randomly drawn without replacement and the null hypothesis is rejected if and only if both are red. Find the probabilities of committing Type I and Type II errors with this criterion.
3. Verify that for the illustration on page 293 the size of the critical region is approximately 0.08.
4. Suppose that in the example on page 292 we had used the decision function

$$d_3(x) = \begin{cases} a_0 & \text{for } x \geq 14 \\ a_1 & \text{for } x < 14 \end{cases}$$

What would have been the probabilities of Type I and Type II error

5. A single observation of a random variable having an exponential distribution is to be used to test the null hypothesis that the mean of the distribution is $\theta = 2$ against the alternative that it is $\theta = 5$. If the null hypothesis is accepted if and only if the observed value of the random variable is less than 3, find the probabilities of Type I and Type II errors.

6. Show that if $\mu_1 < \mu_0$ in the example on page 296, the Neyman-Pearson lemma yields the critical region $\bar{x} \leq \mu_0 - z_\alpha \cdot \frac{1}{\sqrt{n}}$.

7. A random sample of size n from an exponential population is to be used to test the null hypothesis that its parameter is θ_0 against the alternative that its parameter is θ_1, where $\theta_1 > \theta_0$. Use the Neyman-Pearson lemma to find the most powerful critical region of size α, and use the result on page 192 to indicated how to evaluate the constant

8. Use the Neyman-Pearson lemma to indicate how to construct the most powerful critical region of size α to test the null hypothesis that θ, the parameter of a binomial distribution with a given value of n, equals θ_0 against the alternative that it equals $\theta_1 < \theta_0$.

9. If $n = 80$, $\theta_0 = 0.40$, and $\theta_1 = 0.30$, and α is as large as possible without exceeding 0.05, use the normal approximation of the binomial distribution to find the probability of committing a Type II error with the criterion constructed in Exercise 8.

10. A single observation of a random variable having the geometric distribution (3.3.6) is to be used to test the null hypothesis that its parameter equals θ_0 against the alternative that it equals $\theta_1 > \theta_0$. Use the Neyman-Pearson lemma to find the best critical region of size α.

11. Given a random sample of size n from a normal population with $\mu = 0$, use the Neyman-Pearson lemma to construct the most powerful critical region of size α to test the null hypothesis $\sigma = \sigma_0$ against the alternative $\sigma = \sigma_1 > \sigma_0$.

APPLIED EXERCISES

12. An airline wants to test the null hypothesis that 60 percent of its passengers object to smoking inside the plane. Explain under what conditions they would be committing a Type I error and under what conditions they would be committing a Type II error.

THEORETICAL EXERCISES

1. A bowl contains 7 marbles of which θ are red while the others are blue. In order to test the null hypothesis $\theta \leq 2$ against the alternative $\theta > 2$, two of the marbles are randomly drawn without replacement and the null hypothesis is rejected if and only if both are red.
   (a) Find the probabilities of committing Type I errors when $\theta = 0$, 1, and 2.
   (b) Find the probabilities of committing Type II errors when $\theta = 3$, 4, 5, 6, and 7.
   (c) Plot the graph of the power function.
2. Suppose that on page 301 we had wanted to test the null hypothesis $\theta \geq 0.90$ against the alternative hypothesis $\theta < 0.90$ with the use of the decision function $\mathbf{d}_2$ on page 301. Construct the power function by calculating its value for the same values of θ as in the table on page 301.
3. A single observation is to be used to test the null hypothesis that the parameter of the exponential distribution (4.3.2) equals 10 against the alternative hypothesis that it does not equal 10. If the null hypothesis is to be rejected if and only if the observed value is less than 8 or greater than 12, find
   (a) the probability of a Type I error;
   (b) the probabilities of Type II errors when $\theta = 2, 4, 8, 16$, and 20.
   Also plot the graph of the power function.
4. A random sample of size 64 is to be used to test the null hypothesis that the mean of a normal population with the variance $\sigma^2 = 256$ is less than or equal to 40 against the alternative hypothesis that it is greater than 40. If the null hypothesis is to be rejected if and only if the mean of the random sample exceeds 43, find
   (a) the probabilities of Type I errors when $\mu = 37, 38, 39$, and 40;
   (b) the probabilities of Type II errors when $\mu = 41, 42, 43, 44, 45, 46, 47$, and 48.
   Also plot the graph of the power function.

5. The sum of the values obtained in a random sample of size 5 from a Poisson population is to be used to test the null hypothesis that the mean of the population is greater than 2 against the alternative hypothesis that it is less than or equal to 2. If the null hypothesis is to be rejected if and only if the sum of the observations is 5 or less, find
   (a) the probabilities of Type I errors when the mean of the population is 2.2, 2.4, 2.6, 2.8, and 3.0;
   (b) the probabilities of Type II errors when the mean of the population is 2.0, 1.5, 1.0, and 0.5.

   Also plot the graph of the power function. (*Hint:* Use the results obtained in the illustration of Theorem 6.2 on page 193.)
6. Verify the statement on page 304 that 57 heads and 43 tails in 100 flips of a coin does not enable us to reject the null hypothesis that the coin is perfectly balanced (against the alternative that it is not perfectly balanced) at the level of significance $\alpha = 0.05$. (*Hint:* Use the normal approximation of the binomial distribution.)
7. Verify the final step which led to (10.3.6) on page 306.
8. The number of successes in n trials is to be used to test the null hypothesis that the parameter θ of a binomial population equals $\frac{1}{2}$ against the alternative that it does not equal $\frac{1}{2}$.
   (a) Find an expression for the likelihood ratio statistic.
   (b) Use the result of part (a) to show that the critical region of the likelihood ratio test can be written as

$$x \cdot \ln x + (n - x) \cdot \ln (n - x) > k$$

   where x is the observed number of successes.
   (c) Studying the graph of $f(x) = x \cdot \ln x + (n - x) \cdot \ln (n - x)$, its minimum, and its symmetry, show that the critical region of this likelihood ratio test can also be written as $\left| x - \frac{n}{2} \right| > c$, where c is a constant which depends on the size of the critical region.
9. A random sample of size n is to be used to test the null hypothesis that the parameter θ of an exponential population equals θ_0 against the alternative that it does not equal θ_0.
   (a) Find an expression for the likelihood ratio statistic.
   (b) Use the result of part (a) to show that the critical region of the likelihood ratio test can be written as

$$\bar{x} \cdot e^{-\bar{x}/\theta_0} < K$$

(c) Studying the graph of $g(\bar{x}) = \bar{x} \cdot e^{-\bar{x}/\theta_0}$, its maximum, and whether there is any symmetry, show that the critical region of the likelihood ratio test can *not* be written as

$$|\bar{x} - \theta_0| > c$$

but that it can be written as

$$\bar{x} < c_1 \quad \text{or} \quad \bar{x} > c_2$$

where c_1 and c_2 are constants which depend on θ_0 and α.

10. Given a random sample of size n from a normal population with the mean $\mu = 0$, find an expression for the likelihood ratio statistic for testing the null hypothesis $\sigma = \sigma_0$ against the alternative hypothesis $\sigma \neq \sigma_0$. (*Hint:* See Exercise 5 on page 271.)

11. A random sample of size n from a normal population with unknown mean and variance is to be used to test the null hypothesis $\mu = \mu_0$ against the alternative $\mu \neq \mu_0$. Using the simultaneous maximum likelihood estimates of μ and σ^2 obtained in Section 9.4.2, show that the values of the likelihood ratio statistic can be written in the form

$$\lambda = \left(1 + \frac{t^2}{n-1}\right)^{-\frac{n}{2}}$$

where $t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$. Note that the likelihood ratio test can, thus, be based on the t distribution of Section 7.2.4.

12. Independent random samples of size $n_1, n_2, \ldots,$ and n_k from k normal populations with unknown means and variances are to be used to test the null hypothesis $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2$ against the alternative that these variances are not all equal.

(a) Show that under the null hypothesis the maximum likelihood estimates of the means μ_i and the variances σ_i^2 are

$$\hat{\mu}_i = \bar{x}_i \quad \text{and} \quad \hat{\sigma}_i^2 = \sum_{i=1}^{k} \frac{(n_i - 1)s_i^2}{n}$$

where $n = \sum_{i=1}^{k} n_i$, while without restrictions the maximum likelihood estimates of the means μ_i and the variances σ_i^2 are

$$\hat{\hat{\mu}}_i = \bar{x}_i \quad \text{and} \quad \hat{\hat{\sigma}}_i^2 = \frac{(n_i - 1)s_i^2}{n_i}$$

This follows directly from the results obtained in Section 9.4.

(b) Using the results of part (a), show that the likelihood ratio statistic can be written as

$$\lambda = \frac{\prod_{i=1}^{k} \left[\frac{(n_i - 1)s_i^2}{n_i} \right]^{\frac{n_i}{2}}}{\left[\sum_{i=1}^{k} \frac{(n_i - 1)s_i^2}{n} \right]^{\frac{n}{2}}}$$

(c) If $n_1 = 8$, $s_1^2 = 16$, $n_2 = 10$, $s_2^2 = 25$, $n_3 = 6$, $s_3^2 = 12$, $n_4 = 8$, and $s_4^2 = 24$ are the sample sizes and the variances of four independent random samples from four normal populations, use the result of part (b) to calculate $-2 \cdot \ln \lambda$ and then test the null hypothesis stated at the beginning of this exercise. (Note that the number of degrees of freedom for this approximate chi-square test is 3, since $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2$ imposes 3 restrictions on the parameters.)

13. Show that for $k = 2$ the likelihood ratio statistic of Exercise 12 can be expressed in terms of the ratio of the two sample variances and that the likelihood ratio test can, therefore, be based on the F distribution.

14. If 15, 28, 3, 12, 42, 19, 20, 2, 25, 30, 62, 12, 18, 16, 44, 65, 33, 51, 4, and 28 are the values of a random sample from an exponential population, use part (a) of Exercise 9 and Theorem 10.2 to test the null hypothesis that the mean of the population is 15 against the composite alternative that it is not equal to 15. Let α, the size of the critical region, be 0.05.

15. **UNBIASED CRITICAL REGIONS** When we test a simple null hypothesis against a composite alternative, a critical region is said to be *unbiased* if the corresponding power function takes on its *minimum value* at the value of the parameter assumed under the null hypothesis. In other words, *a critical region is unbiased if the probability of rejecting the null hypothesis is least when the null hypothesis is true.* Given a single observation of the random variable x having the density

$$f(x) = \begin{cases} 1 + \theta^2(\frac{1}{2} - x) & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

where $-1 \leq \theta \leq 1$, show that the critical region $x \leq \alpha$ provides an *unbiased* and *uniformly most powerful* critical region of size α for testing the null hypothesis $\theta = 0$ against the alternative hypothesis $\theta \neq 0$.

THEORETICAL EXERCISES

1. Given a random sample of size n from a normal population with the known variance σ^2, show that the null hypothesis $\mu = \mu_0$ can be tested against the alternative $\mu \neq \mu_0$ with the use of a *one-tail criterion* based on the chi-square distribution.
2. Suppose that a random sample from a normal population with the known variance σ^2 is to be used to test the null hypothesis $\mu = \mu_0$ against the alternative hypothesis $\mu = \mu_1$ (with $\mu_1 > \mu_0$), and that the probabilities of Type I and Type II errors are to have the preassigned values α and β. Show that the required size of the sample is given by

$$n = \frac{\sigma^2(z_\alpha + z_\beta)^2}{(\mu_1 - \mu_0)^2}$$

Also use this formula to find n when $\sigma = 12$, $\mu_0 = 28$, $\mu_1 = 32$, $\alpha = 0.05$ and $\beta = 0.01$.
3. Suppose that independent random samples of size n from two normal populations with the known variances σ_1^2 and σ_2^2 are to be used to test the null hypothesis $\mu_1 - \mu_2 = \delta$ against the alternative hypothesis $\mu_1 - \mu_2 = \delta'$, and that the probabilities of Type I and Type II errors are to have the preassigned values α and β. Show that the required size of the samples is given by

$$n = \frac{(\sigma_1^2 + \sigma_2^2)(z_\alpha + z_\beta)^2}{(\delta - \delta')^2}$$

Also use this formula to find n when $\sigma_1 = 12$, $\sigma_2 = 15$, $\delta = 110$, $\delta' = 113$, $\alpha = 0.01$ and $\beta = 0.01$.

APPLIED EXERCISES

4. According to the norms established for an aptitude test, college seniors should average a score of 83.9 with a standard deviation of 8.1. What can we conclude about the seniors attending a large university if 49 of them, randomly selected, averaged 86.4? Use $\alpha = 0.01$.
5. A real estate broker who is anxious to sell a piece of property to a motel chain assures them that during the summer months *on the average* 4,200 cars pass by the property each day. Being suspicious that this figure might be a bit high, the management of the motel chain conducts its own study and obtains a mean of 4,038 cars a day and a standard deviation of 512 cars a day for observations made over 36 days. What can they conclude at the level of significance $\alpha = 0.05$?
6. In 12 test runs over a marked course, a newly-designed motorboat averaged 33.6 seconds with a standard deviation of 2.3 seconds. Assuming that it is reasonable to treat the data as a random sample from a normal population, test the null hypothesis $\mu = 35$ against the alternative $\mu < 35$ at the level of significance $\alpha = 0.05$.
7. The alfalfa yields of six test plots are, respectively, 1.5, 1.9, 1.2, 1.4, 2.3, and 1.3 tons per acre. Use a critical region of size 0.05 to test the null hypothesis $\mu = 1.8$ tons per acre against the alternative hypothesis $\mu \neq 1.8$. (Assume that the yields have normal distribution.)
8. Referring to the numerical example on page 319, the one dealing with the nicotine content of the two kinds of cigarettes, for what range of values of $\bar{x}_1 - \bar{x}_2$ would the null hypothesis have been rejected? Also find the probabilities of committing Type II errors with the given criterion if (a) $\mu_1 - \mu_2 = 1.2$, (b) $\mu_1 - \mu_2 = 1.6$, (c) $\mu_1 - \mu_2 = 2.4$, and (d) $\mu_1 - \mu_2 = 2.8$.
9. A company claims that its light bulbs are superior to those of a competitor on the basis of a study which showed that a sample of 40 of its bulbs had an average "lifetime" of 522 hours (of continuous use) with a standard deviation of 28 hours, while a sample of 30 bulbs made by the competitor had an average "lifetime" of 513 hours (of con-

THEORETICAL EXERCISES

1. Making use of the fact that the chi-square distribution can be approximated with a normal distribution when ν (the number of degrees of freedom) is large, show that for *large samples* from normal populations

$$s^2 \geq \sigma_0^2\left[1 + z_\alpha\sqrt{\frac{2}{n-1}}\right]$$

is an approximate critical region of size α for testing the null hypothesis $\sigma^2 = \sigma_0^2$ against the alternative hypothesis $\sigma^2 > \sigma_0^2$. Also construct corresponding critical regions for testing this null hypothesis against the alternatives $\sigma^2 < \sigma_0^2$ and $\sigma^2 \neq \sigma_0^2$. (See Exercise 8 on page 216.)

2. Making use of the result of Exercise 10 on page 217, show that for large random samples from normal populations, tests of the null hypothesis $\sigma^2 = \sigma_0^2$ can be based on the statistic

$$\left(\frac{s}{\sigma_0} - 1\right)\sqrt{2(n-1)}$$

which has approximately the standard normal distribution.

APPLIED EXERCISES

3. The lifetimes of certain batteries are supposed to have a variance of at least 225 hours. Test the null hypothesis $\sigma^2 = 225$ against the alternative $\sigma^2 > 225$ at $\alpha = 0.05$ if the lifetimes of 20 of these batteries (which constitute a random sample from a normal population) have the sample variance $s^2 = 397$.
4. In a random sample, the weights of 24 Black Angus steers of a certain age have a standard deviation of 238 pounds. Test the null hypothesis $\sigma = 250$ pounds against the two-sided alternative $\sigma \neq 250$ pounds at the level of significance $\alpha = 0.01$.
5. In a random sample, the time which 30 women took to complete the written test for their driver's license had a variance of 6.4 minutes. Test the null hypothesis $\sigma^2 = 8$ against the alternative hypothesis $\sigma^2 < 8$ at the level of significance $\alpha = 0.05$ using
   (a) the method described in the text;
   (b) the method of Exercise 2.
6. Test at the level of significance $\alpha = 0.02$ whether it was reasonable to assume in the example on page 320 dealing with the two kinds of paint that the two (normal) populations have the same variance.
7. Test at the level of significance $\alpha = 0.10$ whether it was reasonable to assume in Exercise 11 on page 323 that $\sigma_1^2 = \sigma_2^2$.
8. Test at the level of significance $\alpha = 0.02$ whether it was reasonable to assume in Exercise 12 on page 323 dealing with the two kinds of rockets that $\sigma_1^2 = \sigma_2^2$.
9. The following are the scores obtained in a personality test by samples of nine married women and nine unmarried women:

Unmarried	88	68	77	82	63	80	78	71	72
Married	73	77	67	74	74	64	71	71	72

Assuming that these data can be looked upon as independent random samples from two normal populations, test the null hypothesis $\sigma_1^2 = \sigma_2^2$ against the one-sided alternative $\sigma_1^2 > \sigma_2^2$ at the level of significance $\alpha = 0.05$. (σ_1^2 and σ_2^2 are, respectively, the variance of the scores of unmarried women and the variance of the scores of married women.)