THE KREIN CONDITION
FOR COHERENT CONFIGURATIONS

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0. Introduction

In the present exposition we will explore some of the fundamental requirements imposed upon the existence of Higman's coherent configurations [5] in the general setting, and upon Biggs' distance regular graphs [2] as a special case. In particular, we will investigate the underlying structure of the Krein condition which was introduced by Scott in [9], as well as the character structure of the centralizer algebras of the configurations. Few of the results are new, although we believe that formula (2.9) is an improvement over (2.4) in [5] and that formula (3.5) has not appeared explicitly in the literature.

1. Preliminaries

The foundation for our study lies in Higman's coherent configurations. As in Higman [5], let X be a nonempty set and $\mathcal{O}$ a set of nonempty binary relations on X so that $\mathcal{O}$ is a subset of the power set $P(X^2)$ of the cartesian square of X. Then we call $(X,\mathcal{O})$ the <u>configuration based on</u> X with $\mathcal{O}$ as its set of <u>basic relations</u>. Denote $n = |X|$ as the <u>degree</u> and $r = |\mathcal{O}|$ as the <u>rank</u> of the configuration. (In case X is a G-space, for some group G, and $\mathcal{O}$ is the totality of G orbits in X^2 we say that $(X,\mathcal{O})$ is <u>afforded by</u> G and call this situation the <u>group case</u>.)

If K is a commutative ring we write $\mathrm{Mat}_K(X)$ for the K-algebra of all matrices, with respect to matrix multiplication, with coefficients in K and having rows and columns indexed by X. For ϕ, $\psi \in Mat_K(X)$ denote $\phi\psi$ as the usual matrix product and $\phi \circ \psi$ as the Hadamard (pointwise) product.

For $H \subseteq X^2$, ϕ_H will stand for the <u>adjacency matrix</u> of the graph (X,H). Hence $\phi_H \in Mat_Z(X)$ and

$$(1.1) \qquad \phi_H(x,y) = \begin{cases} 1 & \text{if } (x,y) \in H, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for a configuration $(X,\mathcal{O})$ denote I_X by $I_X = \{(x,x):x \in X\}$. For $f \in \mathcal{O}$, denote the <u>dual</u> or <u>transpose</u> of f by $f^{\cup} = \{(y,x):(x,y) \in f\}$. By an (f,g) <u>path</u> from x to y we will mean a 3-tuple $(x_0,x_1,x_2) \in X^3$ such that $x_0 = x$, $x_2 = y$ and $(x_0,x_1) \in f$, $(x_1,x_2) \in g$ where f, $g \in \mathcal{O}$. So, a configuration $(X,\mathcal{O})$ is said to be <u>coherent</u> if

(1.2)
i) $\mathcal{O}$ is a partition of X^2,
ii) $f \in \mathcal{O}$, $f \cap I_X \neq \phi$ implies $f \subseteq I_X$,
iii) $f \in \mathcal{O}$ implies $f^{\cup} \in \mathcal{O}$,
iv) for f, g, $h \in \mathcal{O}$ and $(x,y) \in h$, the number a_{fgh} of (f,g) paths from x to y is independent of the choice of x and y.

We will assume (1.2) to be in force throughout.

In our discussion, the submodule Γ of $\mathrm{Mat}_Z(X)$ defined by $\Gamma = \{\phi \in \mathrm{Mat}_Z(X) : \phi|f$ is constant for every $f \in \mathcal{O}\}$ is of utmost importance. Γ-called the <u>adjacency ring</u> of $(X,\mathcal{O})$ - is a free abelian group of rank r having $\mathcal{B} = \{\phi_f : f \in \mathcal{O}\}$ as basis. Denote Γ° as Γ regarded with pointwise operations.

It is easily checked that for every f, $g \in \mathcal{O}$ we have $\phi_f\phi_g = \sum_{h\in\mathcal{O}} a_{fgh}\phi_h$ so that the "intersection numbers" a_{fgh} are really the structure constants of Γ with respect to the basis $\mathcal{B}$. We note also that in the group case $(X,\mathcal{O})$ is automatically a coherent configuration.

We call $(X,\mathcal{O})$ <u>homogeneous</u> if the $n \times n$ identity matrix I is in $\mathcal{O}$. Most of our discussion will utilize the assumption of homogeneity. Keeping this assumption in mind, define the <u>subdegree</u> n_f of $f \in \mathcal{O}$ by $n_f = a_{ff^\cup I}$ and the <u>order</u> of $f \in \mathcal{O}$ by $|f| = nn_f$.

Listed below are some important properties of the intersection numbers.

(1.3) For every f, g, $h \in \mathcal{O}$:

a) $n_f = n_{f^\cup}$

b) $a_{fgh} = a_{g^\cup f^\cup h^\cup}$

c) $\sum_{f\in\mathcal{O}} a_{fgh} = n_{g^\cup}$

d) $\sum_{g\in\mathcal{O}} a_{fgh} = n_f$

e) $a_{fgh^\cup}|h| = a_{hfg^\cup}|g| = a_{ghf^\cup}|f|$.

Also, for later use, we will need the fact that the right regular representation of Γ provides an isomorphism $\phi \xrightarrow{\sim} \hat{\phi}$ of Γ onto a subring $\hat{\Gamma}$, the <u>intersection ring</u>, of $\mathrm{Mat}_Z(\mathcal{O})$ where $\hat{\phi}_f(g,h) = a_{gfh}$ for all f, g, $h \in \mathcal{O}$. The ring $\hat{\Gamma}$ is especially familiar in the context of distance regular graphs.

Properties (1.3) can also be thought of as properties of the set $\hat{\mathcal{B}} = \{\hat{\phi}_f : f \in \mathcal{O}\}$ of $r \times r$ matrices which are a basis of $\hat{\Gamma}$. We will in addition make use of the identity

$$(1.4) \qquad \Delta(\hat{\phi}_g)^t = \hat{\phi}_{g^\cup}\Delta$$

where $\Delta \in \mathrm{Mat}_Z(\mathcal{O})$ is the diagonal matrix given by $\Delta(g,h) = \delta_{gh}n_g$.

We now turn to the representation theory of Γ. In doing this let us replace Γ by the adjacency algebra $C = \mathbb{C}\Gamma$ over $\mathbb{C}$ since we will be considering absolutely irreducible representations. Higman [5] points out that C is semisimple, having the decomposition $C = C_1 \oplus C_2 \oplus \cdots \oplus C_m$ where $C_i = \varepsilon_i C$ and ε_i, $i = 1, \ldots, m$, are the central primitive idempotents of C. Each C_i is isomorphic to a full matrix algebra of degree e_i, say, over $\mathbb{C}$ and

$$(1.5) \qquad r = \sum_{i=1}^{m} e_i^2.$$

Denote by $\zeta_1, \ldots, \zeta_m$ the inequivalent irreducible characters of C where

$$\zeta_i(1) = e_i, \quad \zeta_i(\varepsilon_j) = \delta_{ij}e_i, \qquad (1.6)$$

$$1 \le i, \; j \le m.$$

The vector space $\mathbb{C}X$ has the structure of a (left) module over $\mathrm{Mat}_{\mathbb{C}}(X)$ according to $\phi x = \sum_{y \in X} \phi(y,x)y$, $\phi \in \mathrm{Mat}_{\mathbb{C}}(X)$, $x \in X$, and we call $M = \mathbb{C}X$ the <u>standard module</u> when regarded as a module over C. As such, M admits the decomposition $M = M_1 \oplus \ldots \oplus M_m$ where $M_i = e_i M$ is a direct sum of, say, z_i irreducible isomorphic submodules affording ζ_i, $i = 1, \ldots, m$. Hence, we have

$$n = \sum_{i=1}^{m} z_i e_i. \qquad (1.7)$$

If ζ is the character afforded by M, so that $\zeta(\sigma) = \mathrm{tr}\sigma$ for all $\sigma \in C$, then

$$\zeta = \sum_{i=1}^{m} z_i \zeta_i. \qquad (1.8)$$

Call ζ the <u>standard character</u>; m the <u>reduced rank</u>; $e_1, \ldots, e_m$ the <u>irreducible degrees</u>; and $z_1, \ldots, z_m$ the corresponding <u>multiplicities</u>. Choose notation so that $M_1 = \langle x \rangle_{\mathbb{C}}$ and $e_1 = z_1 = 1$. Call ε_1 the <u>principal idempotent</u>, ζ_1 the <u>principal character</u> and we have

(1.9) $$\zeta_1(\phi_f) = n_f.$$

The algebra C can be completely reduced, i.e., there exists an invertible matrix $U \in \mathrm{Mat}_{\mathbb{C}}(X)$, and even a unitary one if needed, such that for all $\phi \in C$:

(1.10) $$U^{-1}\phi U = \mathrm{diag}(\Delta_1(\phi), \underbrace{\Delta_2(\phi), \ldots, \Delta_2(\phi)}_{z_2}, \ldots, \underbrace{\Delta_m(\phi), \ldots, \Delta_m(\phi)}_{z_m})$$

where $\Delta_1, \ldots, \Delta_m$ are the inequivalent irreducible representations of C affording $\zeta_1, \ldots, \zeta_m$, respectively.

We write $\Delta_\alpha(\phi) = (a^\alpha_{ij}(\phi))$. Define $\varepsilon^\alpha_{ij} \in C$; $i, j = 1, \ldots, e_\alpha$; $\alpha = 1, \ldots, m$ by

(1.11) $$\varepsilon^\alpha_{ij} = U^{-1}\sigma^\alpha_{ij}U \text{ where}$$

$$\sigma^\alpha_{ij} = \mathrm{diag}(0, \underbrace{0, \ldots, 0}_{z_2}, \ldots, \underbrace{\rho^\alpha_{ij}, \ldots, \rho^\alpha_{ij}}_{z_\alpha}, \underbrace{0, \ldots, 0}_{z_{\alpha+1}}, \ldots, \underbrace{0, \ldots, 0}_{z_m})$$

and ρ^α_{ij} is $e_\alpha \times e_\alpha$ having a 1 in the (j,i) entry, 0 everywhere else. It is easily verified that $\{\varepsilon^\alpha_{ij} : i,j = 1, \ldots, e_\alpha, \alpha = 1, \ldots, m\}$ is linearly independent and forms a basis for C.

Also, $|\{\varepsilon^{\alpha}_{ij}: i,j = 1, \ldots, e_{\alpha}, \alpha = 1, \ldots, m\}| = \sum_{\alpha=1}^{m} e_{\alpha}^{2} = r$. Furthermore, $\{\varepsilon^{\alpha}_{ii}: i = 1, \ldots, e_{\alpha}\}$ is a set of orthogonal idempotents for fixed $\alpha = 1, \ldots, m$.

Higman [5] goes on to demonstrate that

$$(1.12) \qquad \varepsilon^{\alpha}_{ij} = z_{\alpha} \sum_{g \in \mathcal{O}} a^{\alpha}_{ij}(\check{\phi}_g)\phi_g,$$

where $\check{\phi}_g = \frac{\phi_{g^\cup}}{|g|}$, and derives the important Schur Relations

$$(1.13) \qquad \sum_{g \in \mathcal{O}} a^{\alpha}_{ij}(\check{\phi}_g) a^{\beta}_{uv}(\phi_g) = \delta_{\alpha\beta}\delta_{iv}\delta_{ju} \frac{1}{z_{\alpha}}$$

and Orthogonality Relations

$$(1.14) \qquad \sum_{g \in \mathcal{O}} \zeta_{\alpha}(\check{\phi}_g)\zeta_{\beta}(\phi_g) = \delta_{\alpha\beta} \frac{e_{\alpha}}{z_{\alpha}}.$$

Finally, we need to include a few comments concerning commutative configurations. We say that the (coherent) configuration $(X, \mathcal{O})$ is <u>commutative</u> if the following equivalent conditions hold:

(1.15) a) Γ is commutative,

b) C is commutative,

c) $a_{fgh} = a_{gfh}$ for all $f, g, h \in \mathcal{O}$,

d) $r = m$,

e) $e_1 = e_2 = \ldots = e_m = 1$.

Commutative configurations are necessarily homogeneous. The map $f \to f^{\cup}$ is <u>trivial</u> if and only if $\phi_f = (\phi_f)^t$ for all $f \in \mathcal{O}$, in which case the configuration is clearly commutative. We will call a configuration <u>trivially paired</u> if $f = f^{\cup}$ for every $f \in \mathcal{O}$.

An important special class of coherent configurations is the class of distance regular graphs of Biggs [2]. We first need some notation and terminology. In a graph Λ, the number of edges traversed in the shortest path joining two vertices u and v is called the <u>distance</u> in Λ between u and v, denoted by $\partial(u,v)$. The <u>diameter</u> d of Λ is given by $d = \max\limits_{u,v \in V\Lambda} \partial(u,v)$ where $V\Lambda$ is the set of vertices of Λ, and we denote $E\Lambda$ as the edge set of Λ. Denote $\Lambda_i(v) = \{u \in V\Lambda : \partial(u,v) = i\}$.

Now, a <u>distance regular graph</u> Λ with diameter d is a regular connected graph of valency k with the following property:

(1.16) There exist natural numbers $b_0 = k$, $b_1, \ldots, b_d$; $c_1 = 1$, $c_2, \ldots, c_d$, such that for each pair of verticies u, v satisfying $\partial(u,v) = j$ we have

i) The number of vertices in $\Lambda_{j-1}(v)$ that are adjacent to u is c_j, $1 \leq j \leq d$,

ii) The number of vertices in $\Lambda_{j+1}(v)$ that are adjacent to u is b_j, $0 \leq j \leq d-1$.

Call $I(\Lambda) = \{k, b_1, \ldots, b_d; 1, c_2, \ldots, c_d\}$ the intersection array of Λ.

We can visualize a distance regular graph as a coherent configuration with the identification $X = V\Lambda$ and $\mathcal{O} = \{f_0 = I, f_1, f_2, \ldots, f_d\}$ where $(x,y) \in f_i$ if and only if $\partial(x,y) = i$, $i = 0, \ldots, d$. Clearly $\mathcal{O}$ partitions X^2; the only $f \in \mathcal{O}$ such that $f \cap I_X \neq \emptyset$ is $f = I$; and $f = f^\cup$ for every $f \in \mathcal{O}$ so that properties i)-iii) of (1.2) hold. Biggs [2] proves that the a_{fgh} satisfy (1.2(iv)), hence every distance regular graph constitutes a coherent configuration. In fact, a homogeneous, trivially paired (and hence commutative) configuration.

Moreover, the a_{fgh} may be identified with $I(\Lambda)$ via

(1.17) $$c_j = a_{f_1 f_{j-1} f_j}; \quad b_j = a_{f_1 f_{j+1} f_j}, \qquad 0 \leq j \leq d,$$

(except for c_0 and b_d which are not defined). Furthermore,

(1.18) $$a_{f_1 f_j f_j} = k - b_j - c_j \qquad 1 \leq j \leq d - 1.$$

The $d + 1$ basic adjacency matrices $A_0 = \phi_{f_0}$, $A_1 = \phi_{f_1}$, $\ldots$, $A_d = \phi_{f_d}$ are given by

$$(1.19) \qquad (A_h)(r,s) = \begin{cases} 1 & \text{if } \partial(r,s) = h, \\ 0 & \text{otherwise.} \end{cases}$$

Biggs [2] simplifies the identity $\phi_f\phi_g = \sum_{h\in\mathcal{O}} a_{fgh}\phi_h$ in that

$$(1.20) \qquad A_1A_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1}, \quad 1 \le i \le d-1,$$

(where $a_i = a_{f_1f_if_i}$, $1 \le i \le d - 1$).

As was already mentioned the intersection ring $\hat{\Gamma}$ will be used later so that in this regard, since $\hat{\Gamma} \cong \Gamma$, we have a similar formula from (1.20) for $\hat{\Gamma}$, namely

$$(1.21) \qquad \hat{A}_1\hat{A}_i = b_{i-1}\hat{A}_{i-1} + a_i\hat{A}_i + c_{i+1}\hat{A}_{i+1},$$

$1 \le i \le d - 1$, where $(\hat{A}_i)_{jk} = a_{f_if_jf_k}$, $i, j, k \in \{0, \ldots, d\}$.

Our main goal is to explore conditions on distance regular graphs and coherent configurations that must be satisfied in order for these systems to exist.

One class of distance regular graphs that has recently come under close scrutiny consists of the generalized polygons of Tits [11, 12]. The feasibility of their existence has been determined in many instances by conditions such as those we will discuss.

We begin with one of the essential conditions - that of Krein.

2. The Krein Condition

Following Higman [5], we use C° to denote the algebra C considered with respect to pointwise addition and multiplication. We will use the set $\{\varepsilon_\lambda : \lambda = 1, \ldots, r\}$ where $\varepsilon_\lambda = \varepsilon_{ij}^\alpha$ if $a_\lambda = a_{ij}^\alpha$ (the a_{ij}^α are assumed to have been listed as $a_1, a_2, \ldots, a_r$ in some order) as our C° basis. We have

$$(2.1) \qquad \varepsilon_\lambda = h_\lambda \sum_{f \in \mathcal{O}} a_\lambda(\check{\phi}_f)\phi_f$$

where $z_\alpha = h_\lambda$, and we note that $h_1, \ldots, h_r$ are positive integers since they are multiplicities of irreducible representations. We will assume that the following three equivalent conditions are in force:

$$(2.2) \quad \begin{aligned} &\text{a)} \quad \Delta_\alpha(\phi)^* = \Delta_\alpha(\phi^*) \quad \text{for every } \phi \in C, \ 1 \leq \alpha \leq m, \\ &\text{b)} \quad \overline{a_\lambda(\phi_f)} = a_{\overline{\lambda}}(\phi_{f^\cup}) \quad \text{for every } f \in \mathcal{O}, \ 1 \leq \lambda \leq r, \\ &\text{c)} \quad \varepsilon_\lambda^* = \varepsilon_{\overline{\lambda}}, \quad 1 \leq \lambda \leq r, \end{aligned}$$

where $a_{\overline{\lambda}} = (a_\lambda)^t$, $\varepsilon_{\overline{\lambda}} = (\varepsilon_\lambda)^t$. If, further, we have $\lambda = \overline{\lambda}$ then ε_λ is a projection. To see this note that (2.2) is equivalent to assuming that the complete reduction of C has been done by a unitary matrix. Therefore $\varepsilon_\lambda = \varepsilon^\alpha_{ij}$ has (see (1.11)) $\varepsilon_\lambda = U^{-1}\sigma_\lambda U$ where U is unitary. Now

$$\begin{aligned}
\varepsilon_\lambda^2 = \varepsilon_\lambda^* \varepsilon_\lambda &= (U^{-1}\sigma_\lambda U)^*(U^{-1}\sigma_\lambda U) \\
&= U^*\sigma_\lambda^*(U^{-1})^*U^{-1}\sigma_\lambda U \\
&= U^*\sigma_\lambda^*\sigma_\lambda U \\
&= U^*\sigma_\lambda U \\
&= U^{-1}\sigma_\lambda U \\
&= \varepsilon_\lambda .
\end{aligned}$$

Therefore, as claimed, $\varepsilon_\lambda^2 = \varepsilon_\lambda$ so that ε_λ is a projection, i.e., an idempotent with eigenvalues 0 and 1. By (2.2c) and the assumption $\lambda = \bar{\lambda}$ we also have that ε_λ is Hermitian. If now $\lambda = \bar{\lambda}$ and $\mu = \bar{\mu}$, then by Schur's Theorem [8] $\varepsilon_\lambda \circ \varepsilon_\mu$ is a positive semi-definite Hermitian matrix with all of its eigenvalues between 0 and 1.

Higman [5] goes on to prove that the structure constants of C° with respect to the basis $\{\varepsilon_\lambda : \lambda = 1, \ldots, r\}$, namely the $c_{\lambda\mu\delta}$ in $\varepsilon_\lambda \circ \varepsilon_\mu = \sum_{\delta=1}^{r} c_{\lambda\mu\delta}\varepsilon_\delta$, have all of their eigenvalues between 0 and 1. As he shows, all of the eigenvalues of

$$c_{\lambda\mu\delta} = h_\lambda h_\mu \sum_{f\in O} \frac{a_\lambda(\phi_f)a_\mu(\phi_f)\overline{a_\delta(\phi_f)}}{|f|^2} \tag{2.3}$$

lie in the interval [0,1]. Equation (2.3) is Higman's version [5] of the Krein condition.

In the special case where ζ_λ, ζ_μ, ζ_δ are all linear characters we get the simplified (and more easily applied) version of (2.3), namely that $0 \leq c_{\lambda\mu\delta} \leq 1$, or

$$(2.4) \qquad 0 \leq \sum_{f \in \mathcal{O}} \frac{\zeta_\lambda(\phi_f)\zeta_\mu(\phi_f)\overline{\zeta_\delta(\phi_f)}}{|f|^2} \leq \frac{1}{h_\lambda h_\mu}.$$

Higman [5] demonstrates that the Krein condition (2.4) can be used to rule out the existence of certain rank 3 coherent configurations. Equation (2.4) also gives an easy proof of the well known condition for generalized quadrangles, namely: $s \leq t^2$, see e.g., Higman [6].

Biggs [3] gives a slight generalization of (2.4) in that he applies Schur's Theorem [8] to the pointwise product of arbitrarily many idempotents ε_λ. His condition is also phrased in the context of distance regular graphs and their intersection arrays rather than the more general coherent configuration setting. The resulting condition is

$$(2.5) \qquad 0 \leq \sum_{f \in \mathcal{O}} \frac{\zeta_{\alpha_1}(\phi_f)\zeta_{\alpha_2}(\phi_f)\cdots\zeta_{\alpha_q}(\phi_f)\overline{\zeta_\beta(\phi_f)}}{|f|^q} \leq \frac{1}{h_{\alpha_1}h_{\alpha_2}\cdots h_{\alpha_q}}$$

which says that all of the eigenvalues of $\varepsilon_{\alpha_1} \circ \varepsilon_{\alpha_2} \circ \cdots \circ \varepsilon_{\alpha_q}$ lie in the interval [0,1]. Biggs [3] proceeds to utilize (2.5) to help decide, via computer searches, upon the feasibility of certain intersection arrays.

One of the first references to the Krein condition (other than by M. G. Krein himself) was made by L. L. Scott [9]. In a later treatment, Scott [10] has used the Krein condition, whose form is nearly identical with (2.4), to prove a result concerning rank 3 permutation groups. One easy application of this result can be made in the case of a rank 3 coherent configuration with trivial pairing. As Higman [5] points out, a primitive rank 3 group of even order affords a coherent configuration of this kind. Suppose G is such a group, then because the adjacency ring Γ is the centralizer in $\mathrm{Mat}_Z(X)$ of the permutation representation of G, see [5], we conclude that in the decomposition $\theta = 1 + \sigma + \rho$ of the permutation character of G: σ, ρ are irreducible centralizer characters. Scott's condition [10] states that

$$\sigma(1) < \tfrac{1}{2}\rho(1)(\rho(1) + 1), \tag{2.6}$$

i.e., $z_2 < \frac{z_3(z_3+1)}{2}$ and symmetrically $z_3 < \frac{z_2(z_2+1)}{2}$ where z_2, z_3 are the multiplicities of characters σ and ρ. In this special case it is certainly easier to apply (2.6) than (2.4)! In fact, Higman [5] lists a few sets of possible values of the parameters for rank 3 trivially paired configurations that fail the Krein condition (2.4) while satisfying all of his other conditions. Yet, in every case listed it is easily checked that (2.6) fails as well. Hence the Krein condition is not only interesting in its own right but also gives rise to easier necessary conditions on the existence of our combinatorial systems.

M. G. Krein [7] discusses the condition in the setting of convex analysis, where he identifies our idempotents ε_λ with the so called zonal kernels of the convex cone of all invariant Hermitian positive semi-definite kernels in C. These zonal kernels constitute the boundary points or vertices of a convex hull within that convex cone.

Hence the Krein condition is really a statement about the size of the convex coefficients used to express an arbitrary element of the convex hull in terms of the vertices. When seen in this frame of reference we can not only gain a deeper appreciation for the reason behind Krein's condition but also, by virtue of the convex analysis involved, have a geometrical realization of the algebras C and C°. We therefore present a new interpretation of the Krein condition which yields, in addition to the satisfying geometric picture, a tighter bound on the structure constants $c_{\lambda\mu\delta}$.

We will assume for simplicity that the adjacency algebra C is commutative (and therefore homogeneous), hence all of its irreducible representations are linear.

Define $\Delta = \{\psi \in C : \psi$ is normed, positive semi-definite, Hermitian$\}$, where "normed" means $\psi(x,x) = 1$ for every $x \in X$, i.e., each matrix in Δ is of the form $I + \phi$ for some $\phi \in C$.

<u>Lemma 2.7</u>; Δ is convex.

<u>Proof</u>. If ϕ_1, $\phi_2 \in \Delta$, $\lambda \in [0,1]$ and $\psi = \lambda\phi_1 + (1 - \lambda)\phi_2$ then we must demonstrate that $\psi \in \Delta$. Now

$$\psi(x,x) = (\lambda\phi_1 + (1 - \lambda)\phi_2)(x,x)$$

$$= \lambda\phi_1(x,x) + (1 - \lambda)\phi_2(x,x) = \lambda + 1 - \lambda = 1$$

for every $x \in X$ so that ψ is normed.

Also, since $\phi_i^* = \phi_i$, $i = 1, 2$, then $\psi^* = \psi$ follows immediately.

If ρ is an eigenvalue of ψ then $\rho = \lambda\alpha_1 + (1 - \lambda)\alpha_2$ where α_i is an eigenvalue of ϕ_i, $i = 1, 2$, since C is commutative. Therefore α_1, $\alpha_2 \geq 0$ implies $\rho \geq 0$ giving us that ψ is positive semi-definite. Hence $\psi \in \Delta$ and Δ is convex. $\square$

Let us assume throughout that $\lambda = \overline{\lambda}$, $\lambda = 1, \ldots, r$, and that (following Higman [5]) we can completely reduce C by a unitary matrix, i.e., that $\varepsilon_\lambda^* = \varepsilon_{\overline{\lambda}} = \varepsilon_\lambda$, $\lambda = 1, \ldots, r$.

The ε_λ are also positive semi-definite since their only eigenvalues are 0, 1.

We now wish to have the ε_λ belong to Δ. In order to accomplish this, they must be normed. Now

$$\varepsilon_\lambda = h_\lambda \sum_{f \in O} \zeta_\lambda(\tilde{\phi}_f)\phi_f$$

(see (2.1)) and for every $x \in X$:

$$\varepsilon_\lambda(x,x) = h_\lambda \zeta_\lambda(\tilde{I})I(x,x) = \frac{h_\lambda \zeta_\lambda(I)}{|I|} = \frac{h_\lambda e_\lambda}{n} = \frac{h_\lambda}{n},$$

(C being commutative implies $e_\lambda = 1$ for every λ).

Hence, as they stand, the ε_λ do not belong to Δ. So let us define the normed elements ξ_λ, $\lambda = 1, \ldots, r$, by $\xi_\lambda = \frac{n}{h_\lambda}\varepsilon_\lambda = n \sum_{f \in O} \zeta_\lambda(\tilde{\phi}_f)\phi_f$, and we now have $\{\xi_\lambda : \lambda = 1, \ldots, r\} \subseteq \Delta$.

Theorem 2.8. The set $\{\xi_\lambda : \lambda = 1, \ldots, r\}$ is the set of vertices for the convex hull Δ.

Proof. First we show that each ξ_μ is a vertex. Suppose there exist ϕ, $\psi \in \Delta$ such that $\xi_\mu = \lambda\phi + (1 - \lambda)$ with $\lambda \in (0,1)$ and $\mu \in \{1, \ldots, r\}$. We wish to show that this is possible only if $\xi_\mu = \psi = \phi$.

Since $\{\varepsilon_\mu : \mu = 1, \ldots, r\}$ is a basis for C then $\{\xi_\mu : \mu = 1, \ldots, r\}$ is also a basis for C. Therefore, for some a_ρ, $b_\rho \in \mathbb{C}$, $\rho = 1, \ldots, r$, we have $\phi = \sum_{\rho=1}^{r} a_\rho \xi_\rho$ and $\psi = \sum_{\rho=1}^{r} b_\rho \xi_\rho$ or

$$\xi_\mu = \lambda \sum_{\rho=1}^{r} a_\rho \xi_\rho + (1 - \lambda) \sum_{\rho=1}^{r} b_\rho \xi_\rho$$

$$= \sum_{\rho=1}^{r} (\lambda a_\rho + (1 - \lambda) b_\rho) \xi_\rho .$$

But, the ξ_ρ, $\rho = 1, \ldots, r$, are linearly independent so that $\lambda a_\mu + (1 - \lambda) b_\mu = 1$ and

$$\lambda a_\rho + (1 - \lambda) b_\rho = 0$$

for every $\rho \neq \mu$.

Now, in the α^{th} irreducible representation of C we have $\zeta_\alpha(\theta) = \lambda_\theta$ is an eigenvalue of θ for $\theta \in C$, so

$$\zeta_\alpha(\phi) = \sum_{\rho=1}^{r} a_\rho \zeta_\alpha(\xi_\rho) = \sum_{\rho=1}^{r} a_\rho \frac{n}{h_\rho} \zeta_\alpha(\varepsilon_\rho)$$

$$= \sum_{\rho=1}^{r} a_\rho \frac{n}{h_\rho} \delta_{\alpha\rho} e_\alpha \qquad \text{(by (1.6))}$$

$$= \frac{a_\alpha n}{h_\alpha} .$$

Hence, necessarily, the $a_\alpha \frac{n}{h_\alpha}$, $\alpha = 1, \ldots, r$, are all eigenvalues of ϕ and similarly the $b_\alpha \frac{n}{h_\alpha}$, $\alpha = 1, \ldots, r$, are all eigenvalues of ψ. Thus, since ϕ and ψ are positive semi-definite, we have a_α, $b_\alpha \geq 0$, $\alpha = 1, \ldots, r$.

Consider $\lambda a_\rho + (1 - \lambda) b_\rho = 0$, $\rho \neq \mu$. We cannot have both a_ρ, $b_\rho > 0$ for we would then have $0 > 0$ since $\lambda \in (0,1)$. So suppose $a_\rho = 0$, then necessarily $b_\rho = 0$, i.e.,

$$\phi = a_\mu \xi_\mu, \qquad \psi = b_\mu \xi_\mu.$$

Now, ϕ and ψ are both normed hence $a_\mu = b_\mu = 1$ which implies that $\phi = \psi = \xi_\mu$ as desired. Thus ξ_μ is a vertex of Δ, $\mu = 1, \ldots, r$. In the other direction we wish to show that any $\gamma \in \Delta$ is a convex combination of the ξ_μ, $\mu = 1, \ldots, r$ which will finish the proof.

Suppose $\gamma \in \Delta$. Since $\Delta \subseteq C$ then γ admits the decomposition $\gamma = \sum_{\mu=1}^{r} c_\mu \xi_\mu$ for some $c_\mu \in \mathbb{C}$, $\mu = 1, \ldots, r$. So we must verify that $c_\mu \geq 0$, $\mu = 1, \ldots, r$, and $\sum_{\mu=1}^{r} c_\mu = 1$. Since $\gamma \in \Delta$ then γ is normed and so are the ξ_μ, $\mu = 1, \ldots, r$, so we get $\sum_{\mu=1}^{r} c_\mu = 1$. By employing the same eigenvalue argument as above, we may also conclude that $c_\mu \geq 0$ for every μ. Hence, γ is expressible as a convex combination of the ξ_μ, $\mu = 1, \ldots, r$, and the proof is complete. $\square$

It is worth observing that under the Hadamard product $\circ$, Δ is a monoid:

1) Associativity is obvious,
2) Closure - by Schur's Theorem [8] $\xi_\mu \circ \xi_\lambda$ is Hermitian positive semi-definite. It is also clear that $\xi_\mu \circ \xi_\lambda$ is normed,
3) Identity-

$$\xi_1 = n \sum_{f \in \mathcal{O}} \frac{\zeta_1(\phi_f \cup)}{|f|} \phi_f = n \sum_{f \in \mathcal{O}} \frac{n_{f\cup}}{|f|} \phi_f \qquad \text{(by (1.9))}$$

$$= \sum_{f \in \mathcal{O}} \frac{n n_f}{|f|} \phi_f \qquad \text{(by 1.3(a)))}$$

$$= \sum_{f \in \mathcal{O}} \phi_f = J$$

the all "1" matrix, which is the identity under $\circ$.

We can now move toward the Krein condition, and here the motivation for the bounds arises from the restriction that convex coefficients must lie in the interval $[0,1]$. Consider $\xi_\mu \circ \xi_\rho$. Since Δ is a monoid then $\xi_\mu \circ \xi_\rho \in \Delta$ and by Theorem 2.8 we have $\xi_\mu \circ \xi_\rho = \sum_{\delta=1}^{r} c'_{\mu\rho\delta} \xi_\delta$ where $\sum_{\delta=1}^{r} c'_{\mu\rho\delta} = 1$, $c'_{\mu\rho\delta} \geq 0$, $\delta = 1, \ldots, r$. In the spirit of Krein, then $0 \leq c'_{\mu\rho\delta} \leq 1$, $\delta = 1, \ldots, r$. Now

$$\xi_\mu \circ \xi_\rho = \frac{n^2}{h_\mu h_\rho}(\varepsilon_\mu \circ \varepsilon_\rho) = \sum_{\delta=1}^{r} c'_{\mu\rho\delta}\xi_\delta = \sum_{\delta=1}^{r} c'_{\mu\rho\delta}\,\frac{n}{h_\delta}\varepsilon_\delta$$

or

$$\varepsilon_\mu \circ \varepsilon_\rho = \sum_{\delta=1}^{r} c'_{\mu\rho\delta}\,\frac{h_\mu h_\rho}{h_\delta n}\varepsilon_\delta = \sum_{\delta=1}^{r} c_{\mu\rho\delta}\varepsilon_\delta$$

so that

$$c'_{\mu\rho\delta} = \frac{h_\delta n}{h_\mu h_\rho}c_{\mu\rho\delta}.$$

But from (2.3) we have

$$c'_{\mu\rho\delta} = h_\delta n \sum_{f\in\mathcal{O}} \frac{\zeta_\mu(\phi_f)\zeta_\rho(\phi_f)\zeta_\delta(\phi_{f^\cup})}{|f|^2}$$

where we have used $\overline{\zeta_\delta(\phi_f)} = \zeta_{\overline{\delta}}(\phi_f\ ) = \zeta_\delta(\phi_{f^\cup})$, the last equality coming from the linearity of all the irreducible representations. Hence the new Krein condition which follows from $0 \leq c'_{\mu\rho\delta} \leq 1$ by convexity is

$$0 \leq \sum_{f\in\mathcal{O}} \frac{\zeta_\mu(\phi_f)\zeta_\rho(\phi_f)\zeta_\delta(\phi_{f^\cup})}{|f|^2} \leq \frac{1}{nh_\delta}, \tag{2.9}$$

which is valid for all μ, ρ, $\delta \in \{1, \ldots, r\}$. By following the same procedure with the convex coefficients of $\xi_{\alpha_1} \circ \xi_{\alpha_2} \circ \ldots \circ \xi_{\alpha_q}$ we can also generalize (2.5), getting

$$(2.9')\ 0 \le \sum_{f \in \mathcal{O}} \frac{\zeta_{\alpha_1}(\phi_f)\zeta_{\alpha_2}(\phi_f)\cdots\zeta_{\alpha_q}(\phi_f)\zeta_\beta(\phi_f)}{f^{\,q}} \le \frac{1}{n^{q-1}h_\beta}.$$

The added strength of this new version (2.9) is best realized when we have trivial pairing, such as in a distance regular graph, in which case h_δ may be replaced by $\max(h_\mu, h_\rho, h_\delta)$ to yield the strongest right hand bound. In this case the right hand bound of (2.4) can also become $(\max(h_\mu, h_\rho, h_\delta))^{-2}$ for its tightest form. Still, (2.9) is better since always $n > h_\delta$, $\delta = 1, \ldots, r$.

Because our picture is phrased in terms of the convex monoid Δ we get the bonus of a geometric interpretation of Δ. Let Π be the convex hull spanned by 0 and $\{\sum \varepsilon_{i_j}\}$ as its vertices, where by $\{\sum \varepsilon_{i_j}\}$ we mean all sums of the form $\varepsilon_{i_1} + \varepsilon_{i_2} + \ldots + \varepsilon_{i_h}$; $i_1, i_2, \ldots, i_h \in \{1, \ldots, r\}$. Therefore, it is easily checked that Π consists only of Hermitian positive semi-definite matrices in C with eigenvalues lying in $[0,1]$.

Let Ω be the convex cone spanned by all positive multiples of the vertices of Π. Again, it is clear that Ω consists only of Hermitian positive semi-definite matrices in C.

In the case of $r = 3$ we can visualize the total scheme via Figure 2.10 below.

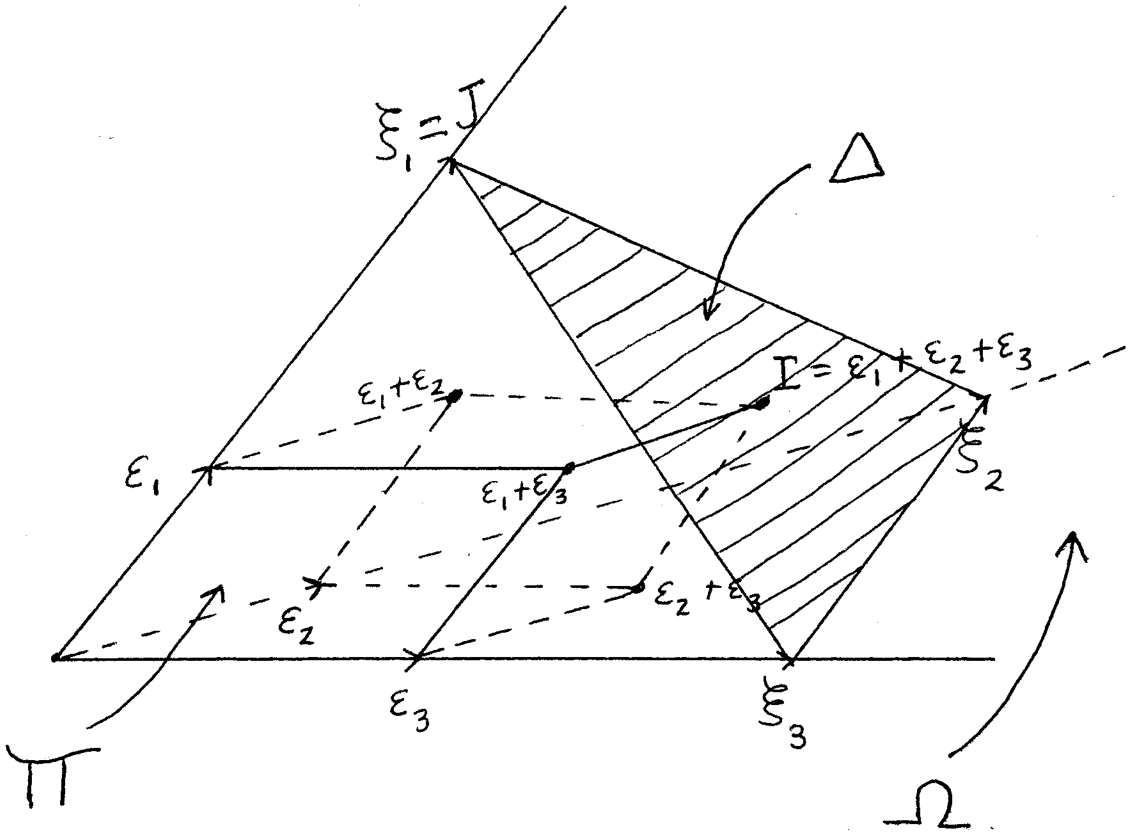


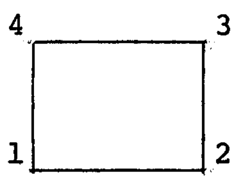
Figure 2.10.

Note that $I = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ is a vertex of the parallelepiped Π and obviously $I \in \Delta$. In fact, we can show that in general $\Pi \cap \Delta = I$. To see this suppose $\phi \in \Pi \cap \Delta$ and also that ϕ is not a vertex of Π. Since $\phi \in \Pi$ then we have the decomposition $\phi = \lambda_0 \cdot 0 + \lambda_1\varepsilon_1 + \ldots + \lambda_r\varepsilon_r$ where $\lambda_i \geq 0$, $i = 0, 1, \ldots, r$, and $\sum_{i=0}^{r} \lambda_i = 1$. So $\phi = \lambda_1 \frac{h_1}{n}\xi_1 + \ldots + \lambda_r \frac{h_r}{n}\xi_r$, and since ϕ is normed then $1 = \lambda_1 \frac{h_1}{n} + \ldots + \lambda_r \frac{h_r}{n}$ giving us $n = \lambda_1 h_1 + \ldots + \lambda_r h_r$. Suppose first that $\lambda_1, \ldots, \lambda_r > 0$ and therefore $\lambda_1, \ldots, \lambda_r < 1$, then $n = \lambda_1 h_1 + \ldots + \lambda_r h_r < h_1 + \ldots + h_r = n$, a contradiction

We must then have $\lambda_1, \ldots, \lambda_k > 0$ for some $k \in \{1, \ldots, r - 1\}$ and $\lambda_{k+1}, \ldots, \lambda_r = 0$ (after suitable renumbering) so that $\lambda_1, \ldots, \lambda_k < 1$. We then have $n = \lambda_1 h_1 + \ldots + \lambda_k h_k < h_1 + \ldots + h_k < n$, another contradiction.

Hence ϕ must be a vertex of Π and clearly the only vertex of Π that lies in Δ is $\phi = I$, showing that $\Pi \cap \Delta = I$ as claimed.

Perhaps it is time for a simple example. Consider the distance regular graph consisting of the square



which we will treat as a coherent configuration with X being the square's vertices. The basic relations $\mathcal{O}$ are f_0: Identity, f_1: adjacency, f_2: non-adjacency. Here $n = 4$, $r = 3$ and we have a trivially paired, commutative, homogeneous configuration. The basic adjacency matrices are

$$\phi_{f_0} = I, \qquad \phi_{f_1} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \qquad \phi_{f_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Using (1.3(d)) we easily compute $n_{f_0} = 1$, $n_{f_1} = 2$, $n_{f_2} = 1$. The character table for the 3 absolutely irreducible inequivalent representations of C is also easily computed, where we use the fact that $\zeta_i(\phi_f)$ is an eigenvalue of ϕ_f for all $f \in \mathcal{O}$, $i = 1, 2, 3$. We get

	ϕ_{f_0}	ϕ_{f_1}	ϕ_{f_2}
ζ_1	1	2	1
ζ_2	1	0	-1
ζ_3	1	-2	1

From (1.14) we also get $z_0 = z_2 = 1$, $z_1 = 2$.

The vertices of Δ are seen to be

$$\xi_1 = J, \quad \xi_2 = I - \phi_{f_2}, \quad \xi_3 = I - \phi_{f_1} + \phi_{f_2}.$$

To get a geometric feel for this configuration, think of I, ϕ_{f_1}, ϕ_{f_2} as "unit vectors" in 3-space and let the "vectors" ξ_1, ξ_2, ξ_3 be written as

$$\xi_1 = (1,1,1), \quad \xi_2 = (1,0,-1), \quad \xi_3 = (1,-1,1)$$

in terms of the "unit vectors". Note that under pointwise multiplication these three "vectors" in fact form a monoid. In this light we get another visualization of Δ below.

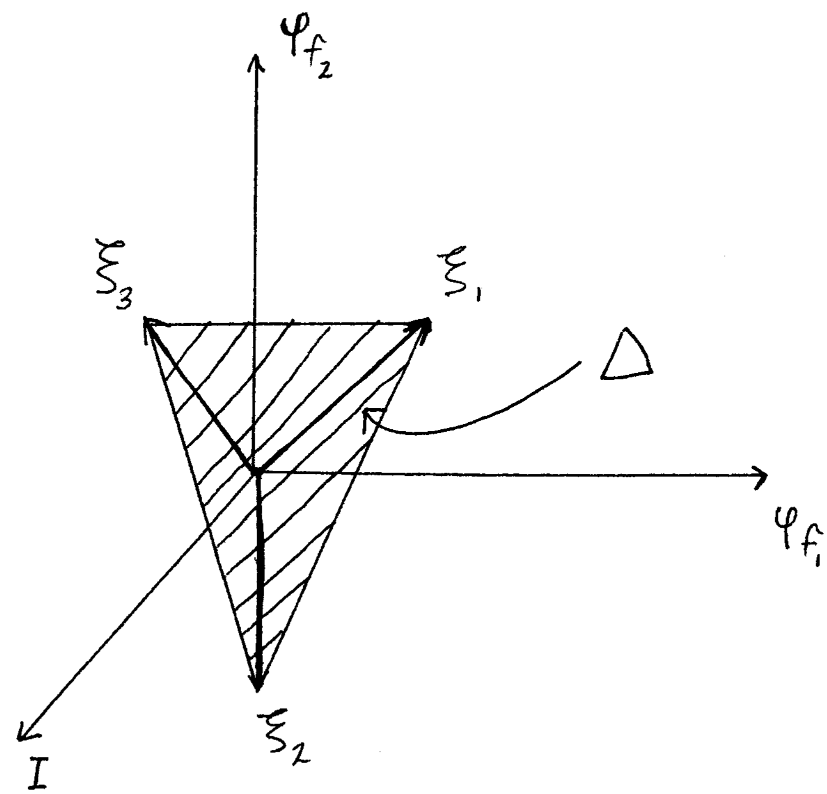


Figure 2.11.

(The author has also investigated the creation of other convex hulls similar to Δ where instead of requiring $\phi(x,x) = 1$ we require $\phi(x,y) = 1$ for all $(x,y) \in f$, some $f \in \mathcal{O}$, $f \neq I$. It turns out that the strongest Krein condition still comes from Δ rather than from a convex hull created by "norming" on ϕ_f, $f \neq I$. Hence, we will not consider those cases.)

We must mention that the tightening of the right end-point bound in our new version of Krein's condition does not really accomplish very much. What we mean is that when the feasibility of a configuration or intersection array is under consideration and the Krein condition is being used, we only need to check the left hand bound. After all, if some $c'_{\mu\rho\delta}$ fails the right bound (i.e., $c'_{\mu\rho\delta} > 1$) then since $\sum_{\delta=1}^{r} c'_{\mu\rho\delta} = 1$ there must be some $c'_{\mu\rho\delta'}$ that is negative to compensate. Hence, in applying Krein's condition it suffices to check the nonnegativity of the appropriate parameters to decide whether the system in question is feasible. But despite the fact that we have tightened the bound to no avail we now know why the condition comes about, via convex analysis, and we have a nice picture of the situation. In addition we can now really state the most useful Krein condition as

$$(2.12) \qquad 0 \leq \sum_{f \in \mathcal{O}} \frac{\zeta_\mu(\phi_f)\zeta_\rho(\phi_f)\zeta_\delta(\phi_{f^\cup})}{|f|^2},$$

since, in application, there is no need for the right hand restriction. (The above form is also the way that Scott [10] states his Krein condition.)

3. Characters in the Centralizer Algebra V(C)

We will develop here a necessary condition on the characters of C and of its centralizer algebra V(C). Let V = V(C) be the centralizer algebra of C,

$$V = \{\psi \in \mathrm{Mat}_{\mathbb{C}} X : \phi\psi = \psi\phi \quad \text{for all} \quad \phi \in C\}.$$

If $\chi_1, \ldots, \chi_m$ are the inequivalent absolutely irreducible characters of V then $\chi_i(1) = z_i$, $i = 1, \ldots, m$. V is semisimple and the central primitive idempotents $\varepsilon_1, \ldots, \varepsilon_m$ of C coincide with those of V. In addition $\chi_i(\psi) = \frac{z_i}{e_i}\zeta_i(\psi)$ for all $\psi \in V \cap C$, $i = 1, \ldots, m$.

As Higman points out [5], every matrix $A \in C$ can be brought to the form

$$(3.1) \qquad A = \mathrm{diag}(A_1 \otimes I_{z_1 \times z_1}, A_2 \otimes I_{z_2 \times z_2}, \ldots, A_m \otimes I_{z_m \times z_m})$$

where $A_i = \Delta_i(A)$, $i = 1, \ldots, m$, is an $e_i \times e_i$ matrix. Consequently, in order for a matrix B to commute with all such A (i.e., to have $B \in V$) it is necessary and sufficient that B can be put into the form:

$$(3.2) \qquad B = \mathrm{diag}(I_{e_1 \times e_1} \otimes B_1, I_{e_2 \times e_2} \otimes B_2, \ldots, I_{e_m \times e_m} \otimes B_m),$$

where B_i is an arbitrary $z_i \times z_i$ matrix over $\mathbb{C}$, $i = 1, \ldots, m$. Combining (3.1) and (3.2) yields

$$(3.3) \qquad BA = \mathrm{diag}(A_1 \otimes B_1, A_2 \otimes B_2, \ldots, A_m \otimes B_m),$$

so that

$$\operatorname{tr}(BA) = \sum_{i=1}^{m} \operatorname{tr}(A_i \otimes B_i) = \sum_{i=1}^{m} \operatorname{tr}(A_i)\operatorname{tr}(B_i) = \sum_{i=1}^{m} \zeta_i(A)\chi_i(B).$$

Now, let $P_g \in V$ be a permutation matrix and consider

$$\operatorname{tr}(P_g\phi_f) = \sum_{i=1}^{m} \zeta_i(\phi_f)\chi_i(P_g) \tag{3.4}$$

for some $f \in \mathcal{O}$. If we multiply both sides of (3.4) by $\zeta_j(\check{\phi}_f)$ and sum over all $f \in \mathcal{O}$ we get:

$$\begin{aligned}
\sum_{f\in\mathcal{O}} \operatorname{tr}(P_g\phi_f)\zeta_j(\check{\phi}_f) &= \sum_{f\in\mathcal{O}} \sum_{i=1}^{m} \zeta_i(\phi_f)\chi_i(P_g)\zeta_j(\check{\phi}_f) \\
&= \sum_{i=1}^{m} \chi_i(P_g) \sum_{f\in\mathcal{O}} \zeta_i(\phi_f)\zeta_j(\check{\phi}_f) \\
&= \sum_{i=1}^{m} \chi_i(P_g)\delta_{ij}\frac{e_i}{z_i} \qquad \text{(by (1.14))} \\
&= \chi_j(P_g)\frac{e_j}{z_j}.
\end{aligned}$$

Hence we have the new found necessary condition on the characters of C and V:

$$\chi_j(P_g) = \frac{z_j}{e_j} \sum_{f\in\mathcal{O}} \operatorname{tr}(P_g\phi_f)\zeta_j(\check{\phi}_f), \quad j = 1, \ldots, m. \tag{3.5}$$

(The motivation for (3.5) is due to an idea by C. W. Curtis in an article yet to appear.) Notice that $P_g\phi_f$ is a matrix whose entries are 0 or 1, and we get a 1 on the diagonal of $P_g\phi_f$ every time P_g sends some k to some ℓ where $(k,\ell) \in f$. Therefore $\mathrm{tr}(P_g\phi_f)$ is the number of pairs (k,ℓ) such that $P_g: k \to \ell$ and $(k,\ell) \in f$.

In the group case, if we take P_g to be the permutation matrix of X afforded by $g \in G$ then $P_g \in V$, since Γ is the centralizer in $\mathrm{Mat}_Z X$ of the permutation representation of G. Hence, for this situation,

$$\mathrm{tr}(P_g\phi_f) = |\{x \in X: (x,x^g) \in f\}|.$$

Also, in this case, $\chi_j(P_g) \in Z$ for all $g \in G$, $j = 1, \ldots, m$, since χ_j is then a permutation character.

One interesting result due to Benson [1] which can also be derived via (3.5) arises from the investigation of the configuration (and distance regular graph) consisting of the proper generalized quadrangle based on X as the vertices. Here $n = (s + 1)(1 + st)$, $r = 3$, and the configuration is trivially paired having character table

	$\phi_{f_0}=I$	ϕ_{f_1}	ϕ_{f_2}
ζ_1	1	$s(t+1)$	s^2t
ζ_2	1	$-1-t$	t
ζ_3	1	$s-1$	$-s$

where the basic relations comprising $\mathcal{O}$ are the Identity f_0, collinearity f_1, and noncollinearity f_2. We have $z_0 = 1$, $z_1 = \frac{s^2(1+st)}{s+t}$, $z_2 = \frac{st(1+t)(1+s)}{s+t}$ (which are integers) $e_0 = e_1 = e_2 = 1$. Benson [1] shows that in the group case $\frac{(t+1)F+L-(1+s)(1+t)}{s+t}$ must be an integer, where F is the number of vertices fixed by a group element g, and L is the number of vertices $x \in X$ such that $(x,x^g) \in f_1$.

An alternative way of arriving at this result comes from (3.5). In fact $F = tr(P_g I)$, $L = tr(P_g \phi_{f_1})$, $n - F - L = tr(P_g \phi_{f_2})$ and we have

$$\chi_3(P_g) = \frac{st(1+t)(1+s)}{s+t}\left[\frac{F}{n} + \frac{L(s-1)}{ns(t+1)} + \frac{(n-F-L)(-s)}{ns^2t}\right]$$

or

$$\chi_3(P_g) = \frac{1}{s+t}(F(t+1) + L - (t+1)(s+1)). \tag{3.6}$$

In the group case, $\chi_3(P_g) \in Z$ so that (3.6) is Benson's result [1] obtained from the general context of the centralizer algebra V.

We conclude this exposition with another application of (3.5). Our example concerns itself with the distance regular graph (coherent configuration) based on a proper generalized hexagon. Here we take X to be the set of vertices of the hexagon and the four basic relations making up $\mathcal{O}$ are

$f_0 = I$, Identity;

f_1, collinearity;

f_2, noncollinear but lying on intersecting lines;

f_3, noncollinear and lying on nonintersecting lines.

We have $n = (s + 1)(1 + st + s^2t^2)$, $r = 4$. The configuration is trivially paired so we wish, in order to employ (3.5), to determine the four linear irreducible representations of C. Hence we must determine all the eigenvalues of ϕ_{f_i}, $i = 0, 1, 2, 3$. From Feit and Higman [4] we learn that

$$z_0 = 1, \quad z_1 = \frac{s^3(1+st+s^2t^2)}{s^2+st+t^2}, \quad z_2 = \frac{(s+1)(t+1)st(1+st+s^2t^2)}{2(1-\sqrt{st}+st)(s+t+\sqrt{st})}, \quad \text{and}$$

$$z_3 = \frac{(s+1)(t+1)st(1+st+s^2t^2)}{2(1+\sqrt{st}+st)(s+t-\sqrt{st})}$$

where st is a square. By making use of (1.3), (1.7), and (1.14) we can deduce via counting arguments that

$$n_I = 1, \quad n_{f_1} = s(t + 1), \quad n_{f_2} = s^2t(t + 1), \quad \text{and} \quad n_{f_3} = s^3t^2.$$

In order to determine the eigenvalues of each ϕ_{f_i} it is easier to examine the intersection matrices $\hat{\phi}_{f_i}$ which are

4×4 matrices. Since $C \cong \hat{C}$, see [2] and [5], then the eigenvalues of a ϕ_{f_i} are precisely those of $\hat{\phi}_{f_i}$ for each $i = 0, 1, 2, 3$.

We already know $\hat{\phi}_{f_0} = I_{4\times 4}$. We can calculate $\hat{\phi}_{f_1}$ via (1.3) and (1.4), getting

$$\hat{\phi}_{f_1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ s(t+1) & s-1 & 1 & 0 \\ 0 & st & s-1 & t+1 \\ 0 & 0 & st & (s-1)(t+1) \end{bmatrix}.$$

To calculate $\hat{\phi}_{f_2}$ we employ (1.21) which tells us that $\hat{\phi}^2_{f_1} = a_{f_1f_1I}I + a_{f_1f_1f_1}\hat{\phi}_{f_1} + a_{f_1f_1f_2}\hat{\phi}_{f_2}$ or, since $a_{f_1f_1I} = s(t+1)$, $a_{f_1f_1f_1} = s - 1$, and $a_{f_1f_1f_2} = 1$, we get

$$\hat{\phi}_{f_2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & st & s-1 & t+1 \\ s^2t(t+1) & st(s-1) & s(t^2+t-1) & (s-1)(t+1)^2 \\ 0 & s^2t^2 & st(s-1)(t+1) & (t+1)(s^2t+t-st-s) \end{bmatrix}.$$

Finally, via (1.3), (1.4) and (1.15) we conclude that

$$\hat{\phi}_{f_3} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & st & (s-1)(t+1) \\ 0 & s^2t^2 & st(s-1)(t+1) & (t+1)(s^2t+t-st-s) \\ s^3t^2 & s^3t^2-s^2t^2 & st(s^2t-st-s+t) & s^3t^2-s^2t^2-t^2+st^2-s^2t+st \end{bmatrix}.$$

We compute the respective eigenvalues and deduce that the character table for this configuration is

	$I_{4\times4}$	ϕ_{f_1}	ϕ_{f_2}	ϕ_{f_3}
ζ_1	1	$s(t+1)$	$s^2t(t+1))$	s^3t^2
ζ_2	1	$-t-1$	$t(t+1)$	$-t^2$
ζ_3	1	$s-1+\sqrt{st}$	$\sqrt{st}(s-1)-s$	$-s\sqrt{st}$
ζ_4	1	$s-1-\sqrt{st}$	$-\sqrt{st}(s-1)-s$	$s\sqrt{st}$

All of the information needed to record (3.5) is present and we have:

(3.7) $$\chi_2(P_g) = \frac{1}{s^2+st+t^2}(\tau_1(s^2-s+1) - \tau_2(s-1) + \tau_3 - (1+st+s^2t^2))$$

$$\chi_3(P_g) = \frac{1}{2\sqrt{st}(s+t+\sqrt{st})}(\tau_1(t+1)(1+\sqrt{st}) + \tau_3 + \tau_2(1+t+\sqrt{st}) - (s+1)(t+1)(1+st+\sqrt{st}))$$

$$\chi_4(P_g) = \frac{1}{2\sqrt{st}(s+t-\sqrt{st}}(\tau_1(t + 1)(\sqrt{st} - 1) - \tau_2(1 + t - \sqrt{st})$$

$$- \tau_3 + (s + 1)(t + 1)(1 + st - \sqrt{st}))$$

where we have written τ_i for $tr(P_g\phi_{f_i})$, $i = 1, 2, 3, 4$. Also, we have suppressed the formula involving χ_1, namely that $\chi_1(P_g) = \frac{1}{n}\sum_{i=1}^{4}\tau_i = 1$ (since χ_1 is the unit character), and have used this latter equation to eliminate τ_4 from the equations of (3.7).

The point here, again, is that we have found a necessary condition on the parameters of a generalized hexagon. In the case where we assume, or know, that a group of collineations is acting on the hexagon, we can best use (3.7) in the sense that each χ_i must be an integer. In this case, we may be able to say more about the sizes of the τ_i and χ_i so that more restrictions on s and t are imposed.

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