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DESIGN CONSTRUCTIBILITY:
STRONGLY REGULAR GRAPHS
AND BLOCK DESIGNS

by

Donald Mark Thompson

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STATEMENT BY AUTHOR

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ABSTRACT

We define an eigengraph of a regular graph Λ as a subgraph Ω for which Ω and $\Lambda \setminus \Omega$ are both regular. It will be shown that Ω is closely related to an eigenvector for the adjacency matrix of Λ. The existence of a particular type of eigengraph in a strongly regular graph G imposes the structure of a balanced incomplete block design in G by means of a partition of the vertices of G into objects and blocks. We investigate the combinatorial and spectral implications for strongly regular graphs that possess this design structure, graphs which we call design constructible.

We give new constructions of some known strongly regular graphs by means of the design approach and in the process are able to determine some interesting subgraph structure for these graphs. We obtain results on the spectral properties of eigengraphs as well as combinatorial properties that they induce in the underlying graphs. We consider the distance regularity and coherency of the eigengraphs arising in design constructible cases and completely classify those instances for which a strongly regular graph is design constructible and has a strongly regular block eigengraph.

A particularly interesting design construction is given for a strongly regular graph on twenty-six vertices. The construction technique enables us to view the graph in terms of the faces and vertices of a regular dodecahedron as well as determine the automorphism group of the graph.

CHAPTER 1

INTRODUCTION

Strongly regular graphs were introduced by Bose (1963) in conjunction with his treatment of partial geometries. Higman (1964) initiated the study of rank 3 permutation groups as automorphism groups for strongly regular graphs and later (1975) developed a general context for incidence structures having a coherent nature. This dissertation deals with the internal structure of strongly regular graphs from both an algebraic and a combinatorial viewpoint. In many instances a strongly regular graph possesses a set of mutually nonadjacent vertices which can be viewed as the set of objects of a balanced incomplete block design. The presence of such an object set yields a partition of the graph's vertices into "object" vertices and "block" vertices whereby an object vertex is "contained in" those block vertices which are adjacent to it. The existence of such a balanced incomplete block design in the vertices coincides with some interesting spectral properties of the graph's adjacency matrix. We will investigate the ramifications of such algebraic and combinatorial structure.

Chapter 2 contains definitions and properties of coherent configurations, distance regular and strongly regular graphs, and block designs. We also include a brief review of some of the known construction techniques for strongly regular graphs. In Chapter 3 the notion of design constructibility (the formalization of our internal structure study) is introduced. The object vertex set is a special type of spectrally related subgraph which we call an eigengraph. In Chapter 4 we develop the spectral and combinatorial properties of those strongly regular graphs possessing the design constructible characteristics. Some examples of non-design constructibility are also provided. Chapter 5 presents new constructions, via block designs, of some known strongly regular graphs. In Chapter 6 we examine some design induced subgraphs and coherent configurations with the help of the spectral analysis of Chapter 4. The results of Chapter 7 provide a geometrically pleasing design construction of a graph on 26 vertices which possesses many symmetry properties due to the presence of the block design.

CHAPTER 2

DEFINITIONS, NOTATIONS, FEASIBILITY, AND CONSTRUCTION TECHNIQUES

2.1. Coherent Configurations

Following Higman (1975) we define the combinatorial entity which provides the general setting for most of our incidence structures. Let X be a nonempty set and $\mathcal{O}$ a nonempty set of binary relations on X, so that $\mathcal{O} \subset P(X^2)$ the power set of the cartesian square on X. We call $(X,\mathcal{O})$ the configuration based on X with $\mathcal{O}$ as its set of adjacency relations. Call $|\mathcal{O}|$ the rank of $(X,\mathcal{O})$. In case X is a G-space for some finite group G and $\mathcal{O}$ is the totality of G orbits on X^2 we say that $(X,\mathcal{O})$ is a group case. For $H \subseteq X^2$, ϕ_H will denote the adjacency matrix of the graph (X,H). Here

(2.1.1)
$$\phi_H(x,y) = \begin{cases} 1 & \text{if } (x,y) \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Set $I_X = \{(x,x): x \in X\}$.

If $(X,\mathcal{O})$ is a configuration and $f \in \mathcal{O}$ denote the dual or transpose of f by $f^{\cup} = \{(y,x):(x,y) \in f\}$.

We call $(X,\mathcal{O})$ trivially paired if $f = f^{\cup}$ for each $f \in \mathcal{O}$. By an (f,g) path from x to y we mean a triple $(x,z,y) \in X^3$ such that $(x,z) \in f$, $(z,y) \in g$, where f, $g \in \mathcal{O}$.

(2.1.2) Definition. A configuration $(X,\mathcal{O})$ is said to be coherent if

(i) $\mathcal{O}$ partitions X^2,

(ii) if $f \in \mathcal{O}$, $f \cap I_X \neq \emptyset$ then $f \subseteq I_X$,

(iii) if $f \in \mathcal{O}$ then $f^{\cup} \in \mathcal{O}$,

(iv) for f, g, $h \in \mathcal{O}$ and $(x,y) \in h$, the number a_{fgh} of (f,g) paths from x to y is independent of the choice of x and y.

Call the parameters a_{fgh} the intersection numbers of $(X,\mathcal{O})$. For coherent $(X,\mathcal{O})$ we have

(2.1.3) $$\phi_f\phi_g = \sum_{h\in\mathcal{O}} a_{fgh}\phi_h \quad \text{for all } f,\ g \in \mathcal{O}.$$

We always assume that our coherent configurations are homogeneous; that is $I_X = I \in \mathcal{O}$. Define the subdegree n_f of $f \in \mathcal{O}$ by $n_f = a_{ff^{\cup}I}$. Some important properties of the intersection numbers are:

(2.1.4) (i) $n_f = n_{f^\cup}$,

(ii) $a_{fgh} = a_{g^\cup f^\cup h^\cup}$,

(iii) $\sum_{f \in \mathcal{O}} a_{fgh} = n_g$,

(iv) $\sum_{g \in \mathcal{O}} a_{fgh} = n_f$.

Denote $\mathrm{Mat}_Z(X)$ as the Z-algebra of all matrices with entries in Z whose rows and columns are indexed by X. Denote $\Gamma = \{\phi \in \mathrm{Mat}_Z(X) : \phi|f$ is constant for each $f \in \mathcal{O}\}$ as the <u>adjacency ring</u> of $(X,\mathcal{O})$. Note that a trivially paired $(X,\mathcal{O})$ yields a commutative Γ. Higman (1975) also showed that (2.1.2)(iv) is equivalent to

(2.1.5) $\left\{\begin{array}{l} F\Gamma \text{ is a ring where } F \text{ is some field} \\ \text{of characteristic } 0. \end{array}\right.$

The right regular representation of Γ provides an isomorphism $\phi \to \hat{\phi}$ of Γ onto $\hat{\Gamma}$ - the <u>intersection ring</u> of $\mathrm{Mat}_Z(\mathcal{O})$. Here we have

(2.1.6) $$\left\{\begin{array}{l} \hat{\phi}_f(g,h) = a_{gfh}, \\ \hat{\phi}_f\hat{\phi}_g = \sum_{h \in \mathcal{O}} a_{fgh}\hat{\phi}_h. \end{array}\right.$$

Note, too, that from (2.1.4)(iii), each $\hat{\phi}_g$ has constant column sum n_g. Regarding $\hat{\Gamma}$, note also that the distinct eigenvalues of $\hat{\phi}$ are precisely those of ϕ, the multiplicities in $\hat{\phi}$ all being 1.

Finally, let $\Delta \in \mathrm{Mat}_Z(\mathcal{O})$ be the diagonal matrix for which $\Delta(g,h) = \delta_{gh} n_g$ and we have

(2.1.7) $$\Delta(\hat{\phi}_g)^T = \hat{\phi}_{g^U}\Delta.$$

2.2. Distance Regular Graphs

A special kind of coherent configuration has been treated by Biggs (1974). Following some terminology, we will define Biggs' distance regular graphs.

In any graph Λ the number of edges traversed in the shortest path joining two vertices u and v is called the distance in Λ between u and v, denoted by $\partial_\Lambda(u,v)$. The diameter of Λ, denoted $\mathrm{diam}(\Lambda)$, is given by $\mathrm{diam}(\Lambda) = \max_{u,v \in V(\Lambda)} (\partial_\Lambda(u,v))$, where $V(\Lambda)$ is the vertex set of Λ. We shall write $u \sim v$ if $\partial_\Lambda(u,v) = 1$ and say that u and v are adjacent; $u \not\sim v$ if $\partial_\Lambda(u,v) > 1$. Paths of length $k - 1$ will be written $\langle u_1, u_2, \ldots, u_k \rangle$ where $u_i \sim u_{i+1}$, $1 \leq i \leq k - 1$.

Denote $\Lambda_i(v) = \{u \in V(\Lambda) : \partial_\Lambda(u,v) = i\}$. Call Λ regular of valence r if $|\Lambda_i(v)| = r$ for all $v \in V(\Lambda)$.

Λ is called connected if there exists a path between any two vertices.

(2.2.1) Definition. A distance regular graph Λ of diameter d is a regular connected graph (of valence r) such that: there exist natural numbers $\sigma_0 = r$, σ_1, σ_2, ..., σ_{d-1}; $\tau_1 = 1$, τ_2, ..., τ_d so that for every vertex pair u, v satisfying $\partial_\Lambda(u,v) = j$ we have

(i) the number of vertices in $\Lambda_{j-1}(v)$ that are adjacent to u is τ_j, $1 \leq j \leq d$,

(ii) the number of vertices in $\Lambda_{j+1}(v)$ that are adjacent to u is σ_j, $0 \leq j \leq d - 1$.

Call $I(\Lambda) = \{r, \sigma_1, ..., \sigma_{d-1}; 1, \tau_2, ..., \tau_d\}$ the intersection array of Λ. From the definitions, it is straightforward to show:

(2.2.2) Theorem. A distance regular graph Λ of diameter d is a homogeneous, trivally paired (hence commutative) coherent configuration of rank $d + 1$ when we identify $X = V(\Lambda)$, $\mathcal{O} = \{I = f_0, f_1, f_2, ..., f_d\}$, where $(u,v) \in f_i$ if and only if $\partial_\Lambda(u,v) = i$, $1 \leq i \leq d$. Moreover, $\tau_j = a_{f_1 f_{j-1} f_j}$, $\sigma_j = a_{f_1 f_{j+1} f_j}$, $0 \leq j \leq d$, (except for τ_0 and σ_d which are not defined), and $a_{f_1 f_j f_j} = r - \sigma_j - \tau_j$, $1 \leq j \leq d - 1$.

2.3. Strongly Regular Graphs

(2.3.1) Definition. A strongly regular graph G, denoted $G = G(n,r,c,\lambda)$, is a distance regular graph of diameter 2. Here $n = |V(G)|$, r is the valence of G, c is the number of triangles on each edge, and λ is the number of paths of length 2 between any nonadjacent pair of vertices.

The facts mentioned in this section may be found in Hestenes and Higman (1971) or Cameron and Van Lint (1975). Simple counting arguments yield:

(2.3.2) $$n = 1 + r + r(r - c - 1)/\lambda.$$

We will write $\phi_{f_1} \equiv A$ for the adjacency matrix of G, having its rows and columns indexed by $V(G)$, so that

(2.3.3) $$A_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly

(2.3.4) $$AJ = JA = rJ$$

where J is the all 1 matrix. Moreover,

$$A^2 = (r - \lambda)I + (c - \lambda)A + \lambda J. \tag{2.3.5}$$

Equations (2.3.4) and (2.3.5) may be taken as an alternate definition of a strongly regular graph, assuming connectedness. Note: some authors allow K_n, the complete graph on n vertices, to be termed strongly regular. We will always assume G has diameter 2.

For any matrix M having distinct eigenvalues $\rho_1, \rho_2, \ldots, \rho_m$ with respective multiplicities $z_1, z_2, \ldots, z_m$ we will write

$$\operatorname{spec}(M) = \begin{bmatrix} \rho_1 & \rho_2 & & \rho_m \\ & & \cdots & \\ z_1 & z_2 & & z_m \end{bmatrix}$$

for the spectrum of M. In particular, it was shown in Hestenes and Higman (1971) that

$$\operatorname{spec}(A) = \begin{bmatrix} r & \rho_2 & \rho_3 \\ & & \\ z_1 = 1 & z_2 & z_3 \end{bmatrix} \tag{2.3.6}$$

where

$$\left\{ \begin{matrix} \rho_2 \\ \rho_3 \end{matrix} \right\} = \frac{1}{2}(c - \lambda \mp \sqrt{(\lambda - c)^2 + 4(r - \lambda)}) \tag{2.3.7}$$

and where

$$
(2.3.8) \qquad \begin{cases} z_2 = \dfrac{\rho_3(n-1)+r}{\rho_3-\rho_2} \\[2ex] z_3 = n - z_2 - 1. \end{cases}
$$

Here $|\rho_2|$, $|\rho_3| \le r$ by the Perron-Frobenius Theorem (see Biggs, 1974). We will write $s = \sqrt{(\lambda-c)^2+4(r-\lambda)}$. As in Hestenes and Higman (1971) we have

$$
(2.3.9) \qquad \begin{cases} 0 < r < n - 1 \\ 0 \le c < r - 1 \\ 0 < \lambda \le r \end{cases}
$$

where $r = n - 1$, $c = r - 1$, and $\lambda = 0$ are excluded since these situations occur only for $G = K_n$.

If $G = G(n,r,c,\lambda)$ is strongly regular then so is its complement $G' = G'(n',r',c',\lambda')$, (which is possibly disconnected), the graph whose adjacency matrix is $J - A - I$. We have

$$
(2.3.10) \qquad \begin{cases} n' = n, \\ r' = n - r - 1, \\ c' = r' - r + \lambda - 1, \\ \lambda' = r' - r + c + 1. \end{cases}
$$

For any graph G and $w \in V(G)$ we will write

(2.3.11)
$$\begin{cases} \Gamma_G(w) = \{u \in V(G): u \sim w\} \\ \Delta_G(w) = \{u \in V(G): u \not\sim w, u \neq w\}. \end{cases}$$

2.4. Block Designs and Independent Sets

In conjunction with the structure of strongly regular graphs we will employ the idea of another incidence structure - a block design.

(2.4.1) Definition. A block design $D = D(v,b,r,k,\lambda)$ is an arrangement of v distinct objects into b sets of equal size k called blocks in such a way that each object appears in r blocks, and every pair of distinct objects appears together in λ blocks.

The well known equations

(2.4.2)
$$\begin{cases} bk = vr \\ \lambda(v - 1) = r(k - 1) \end{cases}$$

may be found in Hall (1967). A complete block design is formed by taking all subsets of size k from a set of size

v to form $b = \binom{v}{k}$ blocks. We will denote the complete block design by K_k^v, having parameters $(v, \binom{v}{k}, \binom{v-1}{k-1}, k, \binom{v-2}{k-2})$.

Objects in a block design D will be denoted $\{o_1, \ldots, o_v\}$ or, when convenient, simply $\{1,2, \ldots, v\}$. Blocks will be written as $\beta = (o_1, o_2, \ldots, o_k)$ and we will alternately view β as a set, for intersection purposes, or as a vertex, for graph purposes. The object-block incidence matrix B of D is that $v \times b$ matrix for which

$$B_{ij} = \begin{cases} 1 & \text{if } o_i \in \beta_j, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, since $(BB^T)_{ij}$ counts the number of blocks containing o_i and o_j, we have

$$BB^T = (r - \lambda)I + \lambda J, \tag{2.4.3}$$

and

$$\operatorname{spec}(BB^T) = \begin{pmatrix} rk & r - \lambda \\ 1 & v - 1 \end{pmatrix}. \tag{2.4.4}$$

Since Fisher's inequality (see Hall, 1967) says

$$(2.4.5) \qquad b \geq v, \qquad (r \geq \lambda)$$

then

$$(2.4.6) \qquad \mathrm{Spec}(B^T B) = \begin{Bmatrix} rk & r - \lambda & 0 \\ 1 & v - 1 & b - v \end{Bmatrix}.$$

We may interpret $(B^T B)_{ij}$ as the number of objects common to blocks β_i and β_j. From definition (2.4.1) and this remark we see that

$$(2.4.7) \qquad \begin{cases} B^T J = kJ \\ BJ = rJ \\ (B^T B)J = J(B^T B) = rkJ \end{cases}$$

for choices of J of appropriate sizes. Hence, by (2.4.3),

$$(2.4.8) \qquad (B^T B)^2 = (r - \lambda)B^T B + k^2 \lambda J.$$

$PG(n,q)$ will denote the projective geometry of dimension n over $GF(q)$, and $EG(n,q)$ the Euclidean geometry of dimension n over $GF(q)$. Each geometry may be viewed as a block design where the geometry's points serve as objects and the geometry's lines serve as blocks (see Hall, 1967).

In order to combine block designs with graphs we need the next definition.

(2.4.9) Definition. An independent set in a graph is a set of mutually nonadjacent vertices.

2.5. Feasibility Conditions

We review the known feasibility conditions for strongly regular graphs; i.e., requirements on the parameters of $G = G(n,r,c,\lambda)$ necessary to its existence. First of all, (2.3.2) must hold. The next, and probably the most important, requirement is the so-called rationality condition which says

(2.5.1) $$z_2 \in Z^+,$$

where z_2 is as in (2.3.8). Equivalent to the rationality condition is the following theorem, which was proved by Hestenes and Higman (1971):

(2.5.2) Theorem. If $G = G(n,r,c,\lambda)$ is a strongly regular graph then one of the following holds:

(i) $r = n - r - 1 = 2\lambda = 2(c + 1)$, or

(ii) $\delta = (\lambda - c)^2 + 4(r - \lambda)$ is a square and

(a) if n is even, $\sqrt{\delta} \mid (2r + (c - \lambda)(n - 1))$

while $2\sqrt{\delta} \nmid (2r + (c - \lambda)(n - 1))$,

(b) if n is odd,

$2\sqrt{\delta} \mid (2r + (c - \lambda)(n - 1))$.

Another known feasibility condition is a bit more subtle. Write the character table of the (commutative) adjacency ring based on $\{I, A, J - A - I\} = \{\phi_{f_0}, \phi_{f_1}, \phi_{f_2}\}$ (using (2.3.6)) as

(2.5.3)

	ϕ_{f_0}	ϕ_{f_1}	ϕ_{f_2}
ζ_1	1	r	$n - r - 1$
ζ_2	1	ρ_2	$-(\rho_2 + 1)$
ζ_3	1	ρ_3	$-(\rho_3 + 1)$

where $\zeta_i(\phi_{f_j})$ is simply an eigenvalue of ϕ_{f_j} with multiplicity z_j. The <u>Krein Condition</u> involves the entries of the character table. See Thompson (1976) or Higman (1975) for more details.

<u>(2.5.4) The Krein Condition</u>. In order for $G = G(n,r,c,\lambda)$ to exist, it is necessary that

$$0 \leq \sum_{i=0}^{2} \frac{\zeta_\alpha(\phi_{f_i})\zeta_\beta(\phi_{f_i})\zeta_\gamma(\phi_{f_i})}{(\zeta_1(\phi_{f_i}))^2}$$

hold for all α, β, $\gamma \in \{1,2,3\}$.

There is another known feasibility condition beyond (2.3.2), (2.5.1), and (2.5.4) in case $\lambda = 1$. Kantor (1978) proved that we must have

$$(2.5.5) \qquad (c+1)\,|\,r, \quad (c+2)\,|\,(\frac{nr}{c+1}).$$

By introducing a few natural assumptions on the structure of a strongly regular graph, we will be able to impose further requirements in the course of the following discussion.

2.6. Some Construction Techniques

Many graph-theoretical constructions have utilized properties of the automorphism groups of graphs (see Hubaut 1975; Hestenes and Higman 1971; Higman 1964 and 1966; Biggs 1971; or Kantor 1978; for examples). We say that $G = G(n,r,c,\lambda)$ is a <u>group case</u> if there is a finite group $\mathcal{G}$ acting in a rank 3 manner (that is $\mathcal{G}_x$ has orbits $\{x\}$, $\Gamma_G(x)$, $\Delta_G(x)$) on $V(G)$. Sophisticated group-theoretic techniques have been used to establish existence or non-existence of certain group cases. In many instances the classical and sporadic simple groups come into play.

For example, Aschbacher (1971) has shown that there can be no rank 3 group for a strongly regular graph $G = G(3250,57,0,1)$, while Kantor (1978) proved the non-existence of the group case when $\lambda = 1$, $c > 0$. Biggs (1971) gives an excellent discussion of the $PSL(n,q)$ groups and the Mathieu groups in terms of the graphs related to them. Generalized quadrangles provide another source of group cases (see Benson, 1970 or Higman, 1964).

The other main source of construction techniques is combinatorial in nature, relying primarily on codes, block designs, difference sets, and latin squares, as well as projective and Euclidean geometries (see, for example, Hubaut (1975) or Cameron and Van Lint (1975)).

The viewpoint taken in this work leans toward combinatorial techniques with a strong spectral flavoring.

CHAPTER 3

DESIGN CONSTRUCTIBILITY AND EIGENGRAPHS

3.1. Motivating Examples

(3.1.1) Example. Consider the unique strongly regular graph $G = G(10,3,0,1)$, known as the Petersen graph, as pictured in Figure 1.

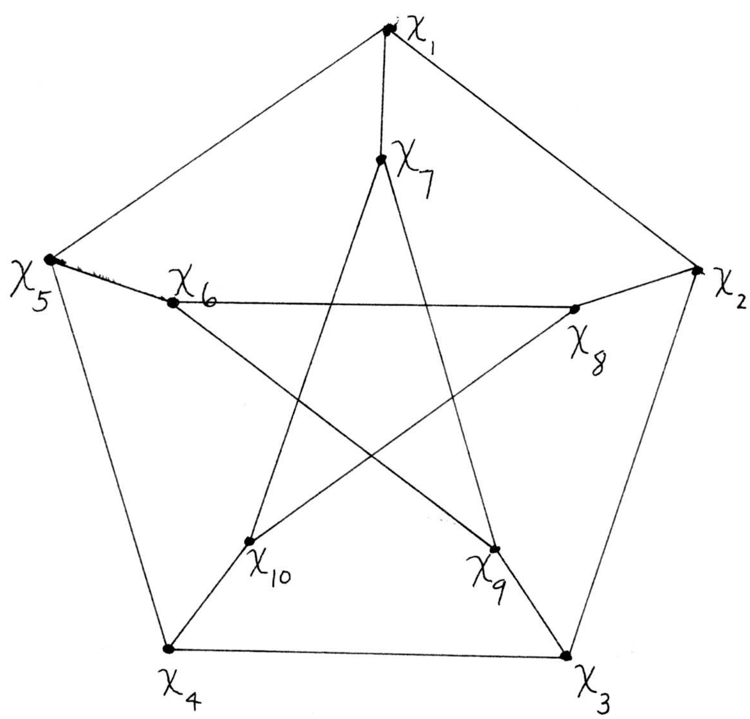

Figure 1. $G = G(10,3,0,1)$, the Petersen Graph.

It is easy to find four mutually nonadjacent vertices, but no more than four, e.g., $\{x_1,x_8,x_9,x_4\}$ is such an independent set. Call these four vertices "objects" and the remaining six vertices "blocks". We will say that a block contains those objects that are adjacent to it in G. For example, block x_7 contains objects x_1 and x_9. Notice that each block contains two objects, each object is contained in three blocks, and each pair of objects is contained in one block. Thus there is a block design $D = D(4,6,3,2,1)$ in the vertices of G. To reverse the process, suppose we are presented with the block design $D = D(4,6,3,2,1)$, (there is but one, namely K_2^4) with objects labeled $\{1,2,3,4\}$ and blocks labeled $\{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$, and are asked to construct $G = G(10,3,0,1)$ from these ten quantities with the requirement that objects be connected to those blocks which contain them. Thus we must decide which vertices are adjacent to a given block. We see that (i,j) is connected to objects i and j and one other vertex which must be a block. That block (ℓ,m) must be disjoint from (i,j) since $c = 0$. This suffices to determine the construction since each block is disjoint from precisely one other block in D.

This simple example illustrates a phenomenon which occurs in many strongly regular graphs. Before we move to the definition that formalizes this situation, let us look at a somewhat more complicated example which will indicate the general problem involved in this approach.

<u>(3.1.2) Example</u>. Consider the strongly regular Hoffman-Singleton graph $G = G(50,7,0,1)$. Hoffman and Singleton (1960) constructed G and established its uniqueness. Following the idea started in the Petersen graph, we seek an independent set in G. Say the independent set contains v vertices, which we call "objects", and call the remaining $50 - v \equiv b$ vertices "blocks". Again, an object is contained in precisely those blocks adjacent to it, thus $r = 7$ tells us that each object is in seven blocks. Assume that the number of objects adjacent to a given block is some number k, independent of the block chosen. Finally, note that each pair of objects is contained in one block since $\lambda = 1$. We have, therefore, a block design $D = D(v,b,7,k,1)$ where $b + v = 50$ and (by (2.4.2)) $bk = 7v$, $v - 1 = 7(k - 1)$. The unique positive integer solution to these three equations is easily seen to be $v = 15$, $b = 35$, $k = 3$. Hence, the familiar design parameters $(15,35,7,3,1)$ appear. With the assumed design structure underlying G, we now attempt to reconstruct G from some design D having the stated parameters.

Example (3.1.2) will be constructed via the design approach in Chapter 5.

3.2. Design Constructibility

We now formalize the structure which the examples above have illustrated.

(3.2.1) Definition. Call a strongly regular graph $G = G(n,r,c,\lambda)$ design constructible (d.c.) if the vertices of G can be partitioned into two sets V and $\mathcal{B}$ such that

(i) V is an independent set,

(ii) $|\{o_i \in V : o_i \sim \beta\}|$ is independent of the choice of $\beta \in \mathcal{B}$.

(3.2.2) Definition. Call a set V of vertices which satisfies (3.2.1)(i) and (ii) an object set.

(3.2.3) Theorem. Let $G = G(n,r,c,\lambda)$ be a design constructible strongly regular graph. Then

(i) V and $\mathcal{B}$ form a block design D;

(ii) the parameters of D are given by $v = r(-\rho_2 - 1)/\lambda + 1$, $b = n - v$, $r = r$, $k = -\rho_2$, $\lambda = \lambda$. Where ρ_2 is the unique negative eigenvalue of the adjacency matrix of G.

Proof. Write $V = \{o_1, o_2, \ldots, o_v\}$, $\mathcal{B} = \{\beta_1, \ldots, \beta_b\}$, where $v = |V|$, $b = n - v = |\mathcal{B}|$. V is an object set, and we shall call the vertices in $\mathcal{B}$ blocks. Define the following incidence relation on $V \cup \mathcal{B}$: $o_i \in \beta_j$ if and only if $o_i \sim \beta_j$ in G. Clearly $o_i \not\sim o_j$, for all i, j. Thus $\Gamma_G(o_i) \subseteq \mathcal{B}$, $1 \leq i \leq v$, so that each object is contained in (is adjacent to) r blocks. Since (3.2.1)(ii) is in force each block contains $k \equiv |\{o_i \in V: o_i \sim \beta_j\}|$ objects. Finally, since nonadjacent G vertices have λ paths between them, any pair of objects is contained in λ blocks. Thus V and $\mathcal{B}$ define a block design, establishing (i). For (ii) we use (2.4.2) which says that $\lambda(v - 1) = r(k - 1)$ and $(n - v)k = vr$. Eliminating v from these equations yields:

$$(3.2.4) \qquad rk^2 + k(r(r - 1) - \lambda(n - 1)) + r(\lambda - r) = 0.$$

Applying (2.3.2) to (3.2.4) gives

$$(3.2.5) \qquad k^2 + k(c - \lambda) - (r - \lambda) = 0.$$

Hence $k = \frac{1}{2}(\lambda - c \pm \sqrt{(\lambda-c)^2+4(r-\lambda)})$, and $k > 0$ demands we take the plus sign so that from (2.3.7) we have

$$(3.2.6) \qquad k = -\rho_2.$$

The stated form for v follows from (2.4.2) and (3.2.6). □

In case G is d.c. by the design D we will say that D affords G and that (G,D) is an affordable pair. $G_{\mathcal{B}}$ will denote $G|\mathcal{B}$ and for $\beta \in \mathcal{B}$ $\Gamma_{G_{\mathcal{B}}}(\beta)$ will denote the set of $r - k$ blocks adjacent to β, $\Delta_{G_{\mathcal{B}}}(\beta)$ denoting the $b - (r - k) - 1$ blocks not adjacent to β.

By virtue of (2.5.2) note that $\rho_2 \in Z$ for every strongly regular graph with the possible exception of self-complementary graphs. There are many strongly regular graphs for which $v \notin Z$. For example, the complement of the Hoffman-Singleton graph is not afforded by any design D since it has $v \notin Z$.

Regarding v, a result of Haemers (1978) is worth mentioning.

(3.2.7) Theorem. Let H be a regular graph on n vertices whose adjacency matrix has eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $(\lambda_1 = \text{valence})$. Then any independent set in H contains at most $\frac{n(-\lambda_n)}{\lambda_1 - \lambda_n}$ vertices.

Applying (3.2.7) to a d.c. $G = G(n,r,c,\lambda)$ gives a bound of $\frac{nk}{r+k}$ on the number of independent vertices. From (2.4.2) and (3.2.3) the bound becomes $\frac{(b+v)k}{r+k} = \frac{v(r+k)}{r+k} = v$, so that in our case an object set has maximal order.

We now cite some instances of G, G' pairs for which both G and G' are d.c. and others for which G is not d.c. while G' is d.c.

<u>(3.2.8) Theorem.</u> Let $T(m)$ $m \geq 4$ denote the strongly regular <u>triangle graph</u> formed by taking as vertices the 2-element subsets of a set of cardinality m and having two vertices adjacent if and only if they are not disjoint. Then $T(m)$ is d.c. for all $m \equiv 0 \pmod 2$ while $T'(m)$ is d.c. for all m.

<u>Proof.</u> The parameters of $T(m)$ are seen to be $n = \binom{m}{2}$, $r = 2(m-2)$, $c = m-2$, $\lambda = 4$. Denote the m-set by $\{1,2, \ldots, m\}$. Clearly $V = \{\{1,2\},\{3,4\}, \ldots, \{m-1,m\}\}$ is an independent set. For any $\{i,j\} \notin V$ we have $\{i,j\}$ adjacent to precisely two vertices in V. Hence $T(m)$ is d.c. for $m \equiv 0(2)$. The resulting design (from (3.2.3)) $D(m)$ has parameters $v = \frac{m}{2}$, $b = \binom{m}{2} - \frac{m}{2}$, $r = 2(m-2)$, $k = 2$, $\lambda = 4$. Note that $D(m)$ is obtained by taking four identical copies of the blocks of $K_2^{m/2}$. The parameters of $T'(m)$ (by (2.3.10)) are $n' = \binom{m}{2}$, $r' = \binom{m-2}{2}$, $c' = \binom{m-4}{2}$, $\lambda' = \binom{m-3}{2}$. (If $m = 4$ then $c' = \lambda' = 0$ and $T'(m)$ consists of three disjoint edges; if $m = 5$ then $c' = 0$ and $T'(5)$ is the Petersen graph.) Clearly $V = \{\{1,2\},\{1,3\}, \ldots, \{1,m\}\}$ is an independent set, and every $\{i,j\} \notin V$ is disjoint from $m-3$ vertices of V. Hence $T'(m)$ is d.c. via the design $D'(m) = K_{m-3}^{m-1}$ having parameters $(m-1, \binom{m-1}{2}, \binom{m-2}{2}, m-3, \binom{m-3}{2})$. □

<u>(3.2.9) Theorem.</u> No strongly regular $G = G(27,10,1,5)$ is d.c., whereas there is a strongly regular $G' = G'(27,16,10,8)$ which is d.c.

<u>Proof.</u> In order that any $G = G(27,10,1,5)$ be d.c. we require (using (3.2.3)) a design D with parameters $(9,18,10,5,5)$. Such a design exists (see Hall, 1967). Set $V = \{1,2,3, \ldots, 9\}$. Let $\beta = (1,2,3,4,5)$ be a typical block of any D having the stated parameters. Then $\Gamma_G(\beta) = \{\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,1,2,3,4,5\}$ must be as in Figure 2.

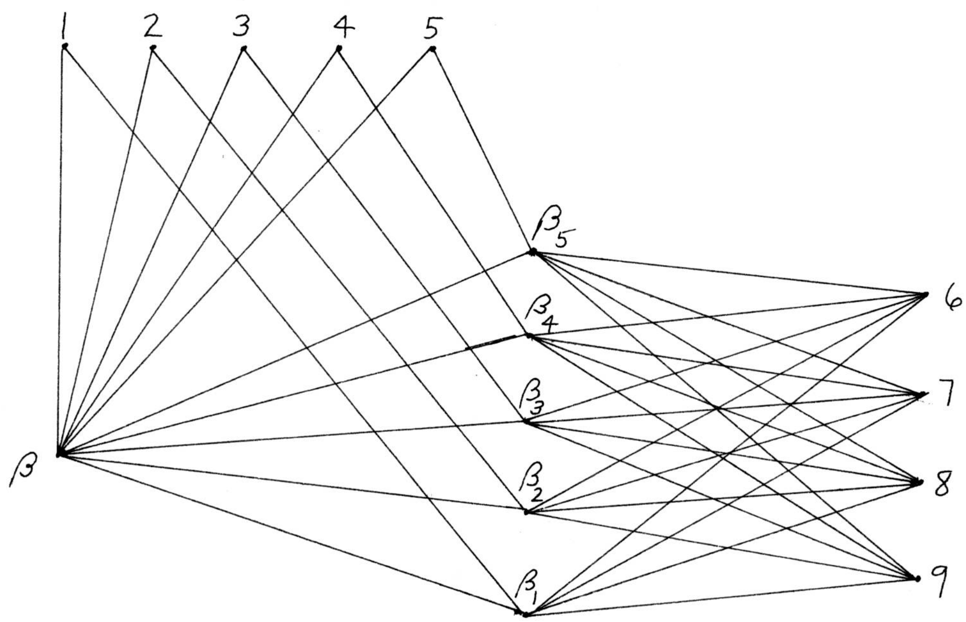

Figure 2. $\Gamma_G(\beta)$.

Since $c = 1$ then $\{\beta_i \cap \beta\} = \{i\}$ and thus $\{\beta_i \cap (V - \beta)\} = \{6,7,8,9\}$, $1 \leq i \leq 5$. From $\lambda = 5$ we see that $\Gamma_{G_B}(\beta_i) \cap \Gamma_{G_B}(\beta_j) = \beta$; $1 \leq i,\ j \leq 5,\ i \neq j$. Hence $\Gamma(\beta_i)$, $i = 1, \ldots, 5$, contributes a total of twenty new blocks beyond β, β_1, β_2, β_3, β_4, β_5. But $b = 18$, a contradiction. Hence G is not d.c.

The G' we speak of is the Schäfli graph (see Cameron et al., 1976), which is represented as follows: the vertices of G' are the vectors

$$\{\vec{e}_i + \vec{e}_j : 1 \leq i,j \leq 6, i \neq j\}$$

$$\cup \{\tfrac{1}{2}\sum_{k=1}^{8} e_k - \vec{e}_i - \vec{e}_7, \tfrac{1}{2}\sum_{k=1}^{8} \vec{e}_k - \vec{e}_i - \vec{e}_8 : 1 \leq i \leq 6\}$$ where $\vec{e}_m$ is the standard basis vector of $\mathbb{R}^8$ which has a 1 in position m, 0 elsewhere. Two vertices are adjacent if and only if their dot product is 1. It is easily checked that $V = \{\vec{e}_1 + \vec{e}_2, \vec{e}_3 + \vec{e}_4, \vec{e}_5 + \vec{e}_6\}$ is an independent set for which all vertices not in V are adjacent to precisely two vertices in V. Hence G is d.c. via $D' = D'(3,24,16,2,8)$ which happens to be eight identical copies of K_2^3. □

We conclude this section by pointing out a somewhat peculiar situation.

(3.2.10) Example. Benson (1970) constructed two strongly regular graphs G, $\bar{G}$ which have the same parameters but are not isomorphic. G is a generalized quadrangle of type (3,3) the vertices of $G = G(40,12,2,4)$ being the points of the quadrangle. $\bar{G}$ is the dual of G. Design construction of G or $\bar{G}$ requires $v = 10$, $k = 4$ by (3.2.3). It is tedious but routine to verify that G has but $\bar{G}$ does not have ten mutually nonadjacent vertices. The remaining requirement (3.2.1)(ii) regarding constant block size is immediate for $\bar{G}$ since $\bar{G}$ is a (rank 3) group case. Hence G is not but $\bar{G}$ is d.c.

3.3. Eigengraphs

We have already seen the close connection between the design parameter k and spec(A). There is an algebraic interpretation of this connection which generalizes to other types of subgraphs (besides the object subgraph) of a strongly regular graph.

Let Λ be any regular graph of valence r on n vertices whose adjacency matrix A_Λ has integer eigenvalues. If Ω is a subgraph of Λ denote by $\vec{\Omega}$ that vector in $(Z_2)^n$ having

(3.3.1) $$\vec{\Omega}_i = \begin{cases} 1 & \text{if vertex } i \text{ is in } \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

<u>(3.3.2) Definition</u>. Call a subgraph Ω of the regular graph Λ an <u>eigengraph</u> of Λ if

(i) Ω is regular of some valence ξ, and

(ii) every vertex in $\Lambda \setminus \Omega$ is adjacent to the same number η of vertices in Ω.

<u>(3.3.3) Theorem</u>. The graph Ω is an eigengraph of a regular (r - valent) graph Λ if and only if

(i) $A_\Lambda \vec{\Omega} = r\vec{\Omega}$, or

(ii) $\xi - \eta$ is an eigenvalue of A_Λ with $\vec{\Omega} - \frac{\eta}{r-(\xi-\eta)}\vec{j}$ a corresponding eigenvector, $\vec{j}$ being the all one vector.

<u>Proof</u>. Note first that (3.3.2) is equivalent to saying

$$A_\Lambda \vec{\Omega} = (\xi - \eta)\vec{\Omega} + \eta\vec{j}, \qquad (3.3.4)$$

since $(A_\Lambda \vec{\Omega})_i$ counts the number of vertices in Ω that are adjacent to a Λ - vertex i. For $\xi = r$, $\eta = 0$ we get $\vec{\Omega} \in \{\vec{j}$ - eigenspace$\}$ and hence (3.3.3)(i) follows from (3.3.4) and conversely.

That (3.3.4) and (3.3.3)(ii) are equivalent (for $\xi - \eta \neq r$) is a matter of using $A_\Lambda \vec{j} = r\vec{j}$ and some matrix multiplication. □

If $\rho \neq r$ is an eigenvalue of A_Λ corresponding to the eigengraph Ω, we will say that Ω is <u>type (ρ)</u>.

An algebraic interpretation of design constructibility is now possible. For $\Lambda = G$, a strongly regular graph (with integer eigenvalues) we see that an object set is merely an eigengraph of type $(-k)$ having $\xi = 0$, $\eta = k$. Thus design constructibility is equivalent to a condition on the structure of the $-k$ eigenspace of A.

From now on we shall write the spectrum of A (the adjacency matrix of a strongly regular $G = G(n,r,c,\lambda)$) as

(3.3.5) $$\mathrm{spec}(A) = \begin{pmatrix} r & -k & s-k \\ \\ 1 & z_2 & z_3 \end{pmatrix},$$

where

(3.3.6) $$s = \sqrt{(\lambda-c)^2+4(r-\lambda)} = 2k - (\lambda - c).$$

Although we will be devoting our attention to object set eigengraphs, it is reasonable to consider other type $(-k)$ as well as type $(s-k)$ eigengraphs. In the process we make another observation:

<u>(3.3.7) Corollary</u>. If Ω is a type (ρ) eigengraph of Λ then $\Lambda \setminus \Omega$ is an eigengraph of type (ρ).

Proof. Note that $(\overline{\Lambda - \vec{\Omega}}) = \vec{j} - \vec{\Omega}$. Say that $A_\Lambda \vec{\Omega} = (\xi - \eta)\vec{\Omega} + \eta\vec{j} = \rho\vec{\Omega} + \eta\vec{j}$, then $A_\Lambda(\vec{j} - \vec{\Omega}) = \rho(\vec{j} - \vec{\Omega}) + (r - \xi)\vec{j}$ and the corollary follows.□

Some questions of interest regarding strongly regular graphs G are:

(3.3.8) (i) Which graphs G have a type (-k) eigengraph?

(ii) Given that G has an object set eigengraph V (of type (-k)), when is $G \setminus V = \mathcal{B}$ also strongly regular?

(iii) When does G have a maximal strongly regular eigengraph of type (s - k)?

(iv) When does $G = \Omega \dot{\cup} (G \setminus \Omega)$, where G and $G \setminus \Omega$ are both type (s - k) or type (-k) and both strongly regular?

This dissertation will, for the most part, treat the ramifications of (3.3.8)(i) when the eigengraph is an object set. The answer to (3.3.8)(ii) is presented in Chapter 6, wherein G is taken to be the union of two strongly regular graphs, the object subgraph being strongly regular in a degenerate (void) sense. For (3.3.8)(iii) the structure of G induced by the presence of a type (s - k) eigengraph is not as well-behaved as with (3.3.8)(i), but we will present a few examples of the phenomenon. For (3.3.8)(iv) we also mention some interesting situations for which $G \setminus \Omega = \Omega$.

Before the examples, we develop more properties of eigengraphs. Let Ω_ρ denote a type (ρ) eigengraph of a regular graph Λ of valence r. Denote by $|\Omega_\rho|$ the number of vertices in Ω_ρ.

(3.3.9) Theorem. Suppose Λ is a regular graph of valence r on n vertices. Let the adjacency matrix A_Λ of Λ have distinct integer eigenvalues $r > \rho_2 > \rho_3 > \dots > \rho_m$. Let the valence of a type (ρ_i) eigengraph Ω_{ρ_i} be denoted ξ_i, $2 \leq i \leq m$. Then

(i) $|\Omega_{\rho_i}| = \dfrac{n\eta_i}{r-\rho_i}$ $2 \leq i \leq m$, where $\eta_i = \xi_i - \rho_i$.

(ii) $|\Omega_{\rho_i} \cap \Omega_{\rho_j}| = |\Omega_{\rho_i}||\Omega_{\rho_j}|/n$, $2 \leq i$, $j \leq m$, $i \neq j$.

Proof. A_Λ is symmetric, hence distinct eigenvalues afford orthogonal eigenvectors. Since $\vec{j}$ is an eigenvector for r then $(\vec{\Omega}_{\rho_i} - \frac{\eta_i}{r-\rho_i}\vec{j},\vec{j}) = 0$, $2 \leq i \leq m$, which reduces to (i) since $(\vec{\Omega}_{\rho_i},\vec{j}) = |\Omega_{\rho_i}|$, $(\vec{j},\vec{j}) = n$. For (ii) we have $(\vec{\Omega}_{\rho_i} - \frac{\eta_i}{r-\rho_i}\vec{j},\vec{\Omega}_{\rho_j} - \frac{\eta_j}{r-\rho_j}\vec{j}) = 0$. Hence, by (i):

$$|\Omega_{\rho_i} \cap \Omega_{\rho_j}| = (\vec{\Omega}_{\rho_i}, \vec{\Omega}_{\rho_j}) = \frac{n\eta_i\eta_j}{(r-\rho_i)(r-\rho_j)} = |\Omega_{\rho_i}||\Omega_{\rho_j}|/n. \quad \square$$

Regarding case (3.3.8)(iii), we cite the following examples:

(3.3.10) Example. Note that the Hoffman-Singleton graph $G = G(50,7,0,1)$ has spectrum $\begin{bmatrix} 7 & -3 & 2 \\ 1 & 21 & 28 \end{bmatrix}$. Any type (2) eigengraph Ω_2 has (via (3.3.9)) 10η vertices with valence $\xi = 2 + \eta$. For Ω_2 to be strongly regular we would need 0 triangles on each edge and one path of length two between nonadjacent vertex pairs. For $c = 0$, $\lambda = 1$ there are but four possible graph sizes, (see Higman (1964)), namely $n = 5$, giving the pentagon; $n = 10$, giving the Petersen graph; $n = 50$, giving the Hoffman-Singleton graph; and $n = 3250$, a graph whose existence is unknown; comprising the so-called Moore graphs. Hence we must choose $\eta = 1$ and get $\Omega_2 = G(10,3,0,1)$, the Petersen graph, (an embedding property which is well known). Applying the same procedure to the Petersen graph we verify that it contains the pentagon as a (type $(s-k)$) eigengraph. Having obtained this sequence of embedded eigengraphs, we naturally ask if the largest Moore graph $G = G(3250,57,0,1)$ could have the Hoffman-Singleton graph as an eigengraph of either type? After computing the spectrum we find that, unfortunately, this is impossible.

Any type $(s - k)$ eigengraph in $G = G(3250,57,0,1)$ would have 65η vertices and valence $\eta + 7$, while a type $(-k)$ eigengraph would have 50η vertices and valence $\eta - 8$.

<u>(3.3.11) Example</u>. The second example of type $(s - k)$ embedding involves an unknown graph $G = G(76,21,2,7)$ having spectrum $\begin{pmatrix} 21 & -7 & 2 \\ 1 & 19 & 56 \end{pmatrix}$. Any eigengraph of type $(s - k)$ requires 4η vertices with valence $\eta + 2$. In order that such an eigengraph E_η be strongly regular it is necessary that E_η have parameters $E_\eta = E_\eta(14\eta, \eta + 2, c_\eta, \lambda_\eta)$ where $c_\eta \leq 2$, $\lambda_\eta \leq 7$. From all the known parameter requirements of 2.3 and 2.4 we find that the permissible strongly regular eigengraphs have parameters:

$$\begin{cases} E_{16} = E_{16}(64,18,2,6) \\ E_{10} = E_{10}(40,12,2,4). \\ E_4 = E_4(16,6,2,2) \end{cases}$$

We can eliminate E_{16} immediately. If E_{16} were a subgraph of G then any vertex $x \in G \setminus E_{16}$ would be adjacent to $\eta = 16$ vertices $\{y_1, \ldots, y_{16}\}$ in E_{16} and five vertices $\{z_1, \ldots, z_5\}$ in $G \setminus E_{16}$. Now $y_i \not\sim y_j$ $1 \leq i, j \leq 16$ $(i \neq j)$ since $c = 2$. Each edge $\langle x, y_i \rangle$ is in two triangles in G. Thus x and y_i are mutually adjacent to two of the z_i's. This requires $16 \cdot 2 = 5q$, where q is the number of y_i vertices adjacent to a z_j. Hence E_{16} is eliminated. No such violation occurs for E_{10} so E_{10} is the maximal possible type (2) eigengraph in G. Interestingly enough, there are constructions known for E_{10} and E_4 (for E_{10} see Higman 1964, and for E_4 see Cameron and Van Lint 1975). It is easily checked that E_4 is an eigengraph of E_{10} and E_4 also contains K_4 as an eigengraph, all of type (s - k). Perhaps it is possible that this sequence of eigengraphs can be extended to include G itself.

Regarding (3.3.8)(iv), we mention two examples wherein $\Omega_\rho = G \setminus \Omega_\rho$ is strongly regular and we defer another example until Chapter 7.

(3.3.12) Example. The Petersen graph can be partitioned into two type (s - k) eigengraphs, namely two pentagons. The Higman-Sims graph $G = G(100,22,0,6)$ (see Gewirtz, 1969) has been shown by Sims (1969) to contain two disjoint Hoffman-Singleton graphs, indeed two type (-k) eigengraphs.

CHAPTER 4

DESIGN INDUCED CONDITIONS

The presence of an object set of size v in a strongly regular graph induces many interesting spectral relationships between the adjacency matrix A of the graph, the block intersection matrix $Y = B^T B$ of the design, and the $b \times b$ submatrix C of A describing block adjacencies. In 4.1 we will determine spec(C) and develop several matrix relationships between Y and C. Several counting formulas will be mentioned in 4.2 regarding the distribution of objects in the blocks of the design. In section 4.3 some of the formulas of 4.2 will be applied to establish the non-design constructibility of a certain graph.

4.1. Spectral and Matrix Properties

Assume that $G = G(n,r,c,\lambda)$ is afforded by $D = D(v,b,r,k,\lambda)$ with design parameters as described in Chapter 3. Index the vertices of G so that the first v vertices $\{o_1, \ldots, o_v\}$ correspond to the objects and the last b vertices $\{\beta_1, \ldots, \beta_b\}$ correspond to the blocks of D. Let B be the $v \times b$ object-block incidence matrix of D (as in 2.4) and denote by C the $b \times b$ matrix for which $C_{ij} = 1$ if $\beta_i \sim \beta_j$, 0 otherwise. We write A as

$$A = \begin{pmatrix} 0 & B \\ B^T & C \end{pmatrix}. \tag{4.1.1}$$

(4.1.2) Theorem. The spectrum of C is

$$\operatorname{Spec}(C) = \begin{pmatrix} r-k & -k & s-k & s-2k \\ 1 & z_2 - v & z_3 - (v-1) & v-1 \end{pmatrix}$$

where z_2, z_3 are as in (2.3.8).

We need a lemma before beginning the proof.

(4.1.3) Lemma. (i) $C^2 = (r-\lambda)I + (c-\lambda)C + \lambda J - Y$,

and

(ii) $BC = (c-\lambda)B + \lambda J_{v\times b}$.

Proof. From (2.3.5) and (4.1.1) we get

$$\begin{pmatrix} BB^T & BC \\ CB^T & C^2 + Y \end{pmatrix} = (r-\lambda)\begin{pmatrix} I_{v\times v} & 0 \\ 0 & I_{b\times b} \end{pmatrix} + (c-\lambda)\begin{pmatrix} 0 & B \\ B^T & C \end{pmatrix} + \lambda J.$$

The lemma follows. □

Proof of (4.1.2). We multiply (4.1.3)(ii) on the left by B^T to obtain, via (2.4.7), (4.1.4) $$YC = (c - \lambda)Y + k\lambda J.$$

Note also that

(4.1.5) $$CJ = JC = (r - k)J.$$

We eliminate Y from (4.1.4) and (4.1.3) to obtain

(4.1.6) $$C^3 + 2(\lambda - c)C^2 + ((\lambda - c)^2 - (r - \lambda))C$$
$$- (r - \lambda)(\lambda - c)I = \lambda J(r - (2k - \lambda + c)).$$

Multiply (4.1.6) by C, then eliminate J from the resulting equation via (4.1.6) to obtain

(4.1.7) $$C^4 - C^3(2(c - \lambda) + r - k)$$
$$- C^2((r - \lambda) - (\lambda - c)^2 - 2(r - k)(c - \lambda))$$
$$- C((r - \lambda)(\lambda - c) - (r - k)(r - \lambda - (\lambda - c)^2))$$
$$+ (r - k)(\lambda - c)(r - \lambda)I = 0.$$

Since $r - \lambda = k(s - k)$, $c - \lambda = s - 2k$ (by (3.2.5) and (3.3.6)) we may rewrite (4.1.7) as

$$(4.1.8) \quad (C - (r - k)I)(C - (-k)I)(C - (s - k)I)$$

$$(C - (s - 2k)I) = 0.$$

Thus we have

$$(4.1.9) \quad \operatorname{spec}(C) = \begin{pmatrix} r - k & -k & s - k & s - 2k \\ & & & \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix}$$

for appropriate (positive integer) multiplicities w_1, w_2, w_3, w_4. Note that

$$(4.1.10) \quad tr(C^e) = w_1(r - k)^e + w_2(-k)^e$$

$$+ w_3(s - k)^e + w_4(s - 2k)^e$$

and that C^e_{ii} counts the number of closed paths of length e starting and ending at β_i. For $e = 0, 1, 2$ the counting is simple. For $e = 3$ we can establish:

$$(4.1.11) \quad C^3_{ii} = c(r - 2k), \quad 1 \leq i \leq b.$$

To see (4.1.11) note that each edge $\langle\beta_i,\beta_j\rangle$ of $G_{\mathcal{B}}$ has $c - Y_{ij}$ triangles on it formed from blocks adjacent to both

β_i and β_j. Also $|\Gamma_{G_{\mathcal{B}}}(\beta_i) \cap \beta_i| = ck$ since, writing $\beta_i = (o_1^i, \ldots, o_k^i)$, each adjacent pair β_i, o_ℓ in G $(1 \leq \ell \leq k)$, has c blocks adjacent to both β_i and o_ℓ^i. Hence

$$C_{ii}^3 = \sum_{\substack{\beta_j \\ \beta_j \sim \beta_i}} (c - Y_{ij}) = c(r - k) - \sum_{\substack{\beta_j \\ \beta_j \sim \beta_i}} Y_{ij}$$

$$= c(r - k) - |\Gamma_{G_{\mathcal{B}}}(\beta_i) \cap \beta_i| = c(r - 2k).$$

Consequently

$$(4.1.12) \qquad \begin{cases} t_r(C^0) = b \\ t_r(C) = 0 \\ t_r(C^2) = b(r - k) \\ t_r(C^3) = bc(r - 2k). \end{cases}$$

Combining (4.1.12) with (4.1.10) and using (2.3.8), (3.2.3)(ii), and (2.4.2) gives (4.1.2). □

Note that (4.1.2) tells us that C has, in general, four distinct eigenvalues and can be viewed as the adjacency matrix of a regular, connected graph (because of the Perron-Frobenius Theorem). In certain cases $G_{\mathcal{B}}$ is disconnected (as when $r - k = -k$, $s - k$, or $s - 2k$) or it reduces to a graph with three distinct eigenvalues, cases we will consider in Chapter 6. The next theorem is useful for construction purposes and will be relied upon in Chapters 5 and 7.

(4.1.13) Theorem. Assuming that G is connected $G(n,r,c,\lambda)$ is afforded by $D = D(v,b,r,k,\lambda)$ if and only if

(i) (2.4.3) holds,

(ii) $\Gamma_{G_{\mathcal{B}}}(\beta)$ contains c repetitions of the k objects in β and λ repetitions of the remaining $v - k$ objects of D, for all $\beta \in \mathcal{B}$, and

(iii) the number of blocks adjacent to both β_i and β_j is

(a) $r - k$ if $i = j$,

(b) $c - Y_{ij}$ if $\beta_i \sim \beta_j$,

(c) $\lambda - Y_{ij}$ if $\beta_i \not\sim \beta_j$, for all i, j.

Proof. For (ii) notice that $(BC)_{ij}$ counts the number of repetitions of object o_i in $\Gamma_{G_{\mathcal{B}}}(\beta_j)$, hence (ii) is equivalent to (4.1.3)(ii). Clearly (iii) is equivalent to (4.1.3)(i). Thus the theorem says that (G,D) is an affordable pair if and only if (4.1.3) and (2.4.3) hold. certainly (4.1.3) and (2.4.3) are necessary for design constructibility.

For sufficiency observe that we need only establish (2.3.4) and (2.3.5) when A is written as in (4.1.1). But (4.1.3) and (2.4.3) give us (2.3.5), while the row (column) sum of r for A follows directly from (4.1.13)(ii), (iii) and (2.4.3). □

A situation that utilizes (4.1.13) in the negative sense is

<u>(4.1.14) Corollary</u>. No graph $G = G(16,5,0,2)$ is design constructible.

<u>Proof</u>. In order for a design D to afford $G(16,5,0,2)$ we require (via (3.2.3)) the design parameters (6,10,5,3,2). Hall (1967) established the uniqueness of such a design. We will represent the blocks of D as

$$\{(1,2,3),(1,2,5),(1,4,6),(1,5,6),(1,3,4),\\ (3,4,5),(2,3,6),(2,4,5),(3,5,6),(2,4,6)\}. \tag{4.1.15}$$

Note that every pair of distinct blocks intersects in one or two objects. Hence it is impossible to satisfy (4.1.13)(ii) on the $r - k = 2$ blocks adjacent to a given block. Thus $G = G(16,5,0,2)$ (although it has been constructed by other means, Biggs (1971)) is not design constructible. □

4.2. Counting Conditions

When particular pairs (G,D) are under consideration there are other conditions on blocks than those of 4.1

that can often be applied. As evidenced by (4.1.13), the distribution of the intersections of block pairs plays an important role. The simplest kinds of designs (regarding block intersection) are the symmetric designs having $v = b$, $k = r$ and each block pair intersecting in λ objects. We discuss that situation next.

<u>(4.2.1) Theorem</u>. The only strongly regular graphs afforded by symmetric designs are the complete bipartite graphs $K_{r,r}$.

<u>Proof</u>. Suppose $G = G(2v,r,c,\lambda)$ is afforded by $D = D(v,v,r,r,\lambda)$. From (2.3.2) we obtain $r(r - c - 1) = r(r - 1) + \lambda(v - r)$. Using (2.4.2) we get $\lambda(v - r) = -rc$. Hence, via (2.4.5) we discern that $c = 0$, $v = r$ and hence (by (2.4.2) again) we have $v = r = \lambda$. Thus $G = G(2r,r,0,r)$, which is only possible for $G = K_{r,r}$. Also we see that D consists of a block $\beta = (1,2, \ldots, r)$ repeated r times. See Figure 3. □

We must contend, therefore, with designs having more than one intersection order. The following result, due to Stanton and Sprott (1964), will be helpful.

<u>(4.2.2) The Stanton-Sprott Equations</u>. In a design $D = D(v,b,r,k,\lambda)$ distinguish a block β and set $m_i = |\{\bar{\beta} \in \mathcal{B}\setminus\beta : |\bar{\beta} \cap \beta| = i\}|$, $0 \leq i \leq k$. Then

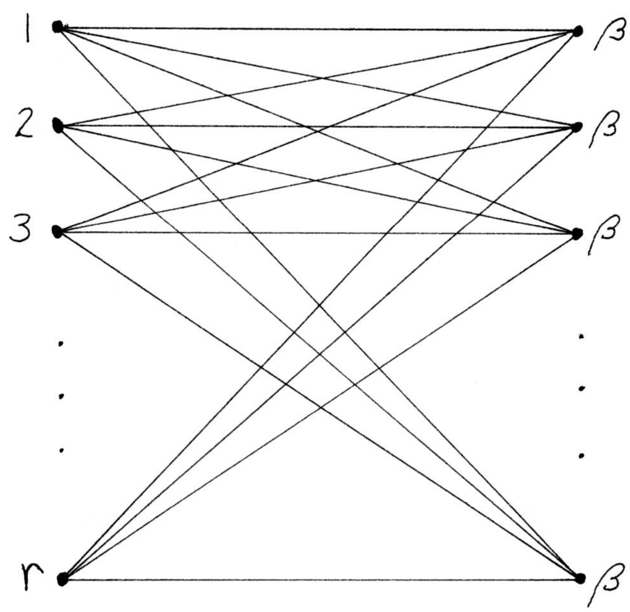

Figure 3. $K_{r,r}$.

(i) $\sum_{i=0}^{k} m_i = b - 1,$

(ii) $\sum_{i=1}^{k} im_i = k(r - 1),$ and

(iii) $\sum_{i=1}^{k} i^2 m_i = k(r - 1) + k(k - 1)(\lambda - 1).$

We will denote by $\{m_0, m_1, \ldots, m_k\}_\beta$ the <u>intersection distribution</u> for block β with the m_i's as described in (4.2.2). It is certainly possible that different blocks have different intersection distributions. This is not the case, however, when $\lambda = 1$ because (4.2.2) reduces to

$m_1 = k(r-1)$, $m_0 = b - 1 - m_1$, $m_i = 0$, $i > 1$ for every block. There are more general situations for which the intersection distribution is invariant.

(4.2.3) Theorem. Let D be a design for which Y has at most three different off-diagonal values, then every block of D has the same intersection distribution.

Proof. First suppose there are three different values, say $i_1 < i_2 < i_3$, then for any block we have (via (4.2.2))

$$m_{i_1} + m_{i_2} + m_{i_3} = b - 1$$

$$i_1 m_{i_1} + i_2 m_{i_2} + i_3 m_{i_3} = k(r-1)$$

$$i_1^2 m_{i_1} + i_2^2 m_{i_2} + i_3^2 m_{i_3} = k(r-1) + k(k-1)(\lambda - 1).$$

The coefficient matrix for this system has a Vandermonde determinant which does not vanish since $i_1 < i_2 < i_3$. Hence there is but one solution $(m_{i_1}, m_{i_2}, m_{i_3})$ to the system.

In case Y has only two different off-diagonal values, then take $i_1 = i_2$ above and check that m_{i_1}, m_{i_3} are uniquely determined. For only one off-diagonal value, D is symmetric. Hence the theorem. □

For the case when we have only two intersection orders in D, D is called quasi-symmetric. Cameron and Van Lint (1975) discuss the use of such designs for constructing strongly regular graphs. In such a case the blocks comprise all of the vertices and two blocks are said to be adjacent if they intersect in a prescribed number of objects. In Chapters 5, 6, 7 we will see other uses of quasi-symmetric designs.

Many times the parameters of D are of the form $(v,b,r,k,\lambda) = (v,\alpha b',\alpha r',k,\alpha\lambda')$ where $\alpha \geq 2$. When such design parameters are encountered in constructions, it is reasonable to ask whether the underlying design $D' = D(v,b',r',k,\lambda')$ can be used to obtain D by merely taking α copies of the blocks of D'. If this is the case, we say that D is a multiple design of multiplicity α.

We state a few propositions regarding the use of designs having large orders of block intersection.

(4.2.4) Proposition. Suppose that $D = D(v,b,r,k,\lambda)$ affords $G = G(n,r,c,\lambda)$ and that D has a block β with $m_j > 0$, $j > \lambda$. Then β must be connected to all blocks $\bar{\beta}$ for which $|\beta \cap \bar{\beta}| = j > \lambda$.

Proof. Suppose $|\beta \cap \bar{\beta}| = j > \lambda$. Having $\partial_G(\beta,\bar{\beta}) = 2$ would entail more than λ paths between β and $\bar{\beta}$ since $j > \lambda$. Hence $\beta \sim \bar{\beta}$. □

(4.2.5) Proposition. Suppose $D = D(v,b,r,k,\lambda)$ is a design having α blocks that intersect pairwise in $j > \lambda$ objects, and that D affords G. Then $\alpha - 2 + j \leq c$.

Proof. By (4.2.4) the α blocks described are adjacent. Hence these α blocks $\beta_1, \beta_2, \ldots, \beta_\alpha$ form a K_α subgraph in G. Since each edge $\langle \beta_i, \beta_\ell \rangle$ $1 \leq i, \ell \leq \alpha$ has j triangles formed from the j objects common to β_i and β_ℓ and $\alpha - 2$ further triangles from the K_α subgraph, it follows that $\alpha - 2 + j \leq c$. □

(4.2.6) Proposition. If $D = D(v,b,r,k,\lambda)$ affords G, and D has some block β with $m_j > 0$, $j > c$, then β is not adjacent to any $\bar{\beta}$ for which $|\beta \cap \bar{\beta}| = j > c$.

Proof. For β, $\bar{\beta}$ as described, $\beta \sim \bar{\beta}$ would mean there were j triangles on $\langle \beta, \bar{\beta} \rangle$. Because $j > c$ we must have $\beta \not\sim \bar{\beta}$. □

Examples that apply the ideas of these propositions are given by the strongly regular graphs having $c = 0$.

(4.2.7) Theorem. If $G = G(n,r,0,\lambda)$ is afforded by $D = D(v,b,r,k,\lambda)$ then $m_j = 0$, $j > \lambda$, for all blocks of D.

Proof. Apply (4.2.4), (4.2.6). □

In particular, no multiple designs are allowed in the construction of graphs having $c = 0$, $r > \lambda$ since $k = \frac{1}{2}(\lambda + \sqrt{\lambda^2+4(r-\lambda)}) > \lambda$. An example of this situation will be seen in Chapter 6 where a multiple design affords the graph $G = G(56,10,0,2)$, which was constructed by Gewirtz (1969).

Regarding design construction of G, notice that the existence of an independent set of size v in G is equivalent to the existence of K_v as a subgraph of G'.

<u>(4.2.8) Theorem</u>. If $G = G(n,r,c,\lambda)$ is afforded by $D = D(v,b,r,k,\lambda)$ then $c' \geq v - 2$, where c' is the number of triangles on each edge of G'.

<u>Proof</u>. As mentioned, we have K_v as a subgraph of G'. Each edge of the subgraph has $v - 2$ triangles, whence $c' \geq v - 2$. □

The case for which $c' = v - 2$ is of some interest.

<u>(4.2.9) Theorem</u>. If $D(v,b,r,k,\lambda)$ affords $G(n,r,c,\lambda)$ with $v - 2 = c'$ then either

(i) $G = K_{k,k,\ldots,k}$, the complete $(\frac{r}{k} + 1)$ - partite graph,

or

(ii) $G = G(v(1 + \alpha(v - 1)),\alpha(v - 1)^2,\alpha(v - 2)^2 + (\alpha - 1)(v - 1),\alpha(v - 2)(v - 1))$,

$G' = G'(v(1 + \alpha(v - 1)),(\alpha + 1)(v - 1), v - 2,\alpha + 1)$, and D is $\alpha(v - 1)$ copies of K^{v}_{v-1}; where $\alpha = \frac{\lambda}{(v-2)(v-1)} \le (v - 1)^2$, $(v - 1)^2\left(\frac{\alpha(v-1)+1}{(\alpha+v-1)}\right) \in Z^{+}$.

Proof. From (2.3.10) and the fact that $n = b + c' + 2$ we have $c' = b + c' + \lambda - 2r$, so that $b + \lambda = 2r$. Using (2.4.2) we obtain $k^2 - k(3 + 2c') + (c' + 1)(c' + 2) = 0$, so either $k = c' + 1 = v - 1$ or $k = c' + 2 = v$. In the latter case we obtain, by (2.4.2), $b = r = \lambda$ and $v = k$. The rationality condition (2.5.1) entails $k|r$. Biggs (1971) has proved the uniqueness of such a $G = G(n,r,c,r)$ graph, namely G is $K_{k,k,\ldots,k}$, the complete $(\frac{r}{k} + 1)$ - partite graph. The design here consists of one block repeated r times and one may easily verify the (trivial) design construction of G. For $k = v - 1$ we use (2.4.2) to obtain $r = \lambda + \frac{\lambda}{c'}$. Since (by (2.3.10)) we have $c' = r' - r + \lambda - 1$ $= b + c' - 2r + \lambda$, it follows that $b = \lambda + \frac{2\lambda}{c'}$. Also $c = \lambda + \frac{\lambda}{c'} - 1 - c' - \frac{\lambda}{c'+1}$ follows from (2.3.2). Since $c'(c' + 1)|\lambda$ we write $\alpha = \frac{\lambda}{c'(c'+1)} = \frac{\lambda}{(v-2)(v-1)}$ and we obtain the indicated parameters of G, G'. For D we get $D = D(v,\alpha(v - 1)v,\alpha(v - 1)^2,v - 1,\alpha(v - 2)(v - 1))$. Clearly D is obtained (uniquely) from $\alpha(v - 1)$ copies of K^{v}_{v-1}. The Krein and rationality conditions (2.5.1), (2.5.4) yield the requirements on α and v as stated in (4.2.9)(ii). □

Note that for fixed c' there are only finitely many feasible graphs having $c' = v - 2$. Note, too that G' has the same parameters as a generalized quadrangle of type $(v - 1,\alpha)$ when considered as a strongly regular graph on its points. In this case the assumption of design constructibility of G is no surprise since the presence of a K_v subgraph in G' amounts to a line with v points on it. For small values of α and c' the graphs are easy to handle and the number of design copies is manageable. In Chapter 5 we will present the constructions for $\alpha = 1$.

We conclude this section with an interesting example.

(4.2.10) Example. Take $c' = 2$ in (4.2.9) and obtain the following permissible cases:

α	G	G'
1	(16,9,4,6)	(16,6,2,2)
3	(40,27,18,18)	(40,12,2,4)
5	(64,45,32,30)	(64,18,2,6)
6	(76,54,39,36)	(76,21,2,7)
9	(112,81,60,54)	(112,30,2,10) .

Constructions are known for every G' case except $\alpha = 6$, viz. generalized quadrangles of type $(3,\alpha)$ (see Payne, 1973). It is a routine matter to verify the presence of 3α copies of K_3^4 in each known case. Again we encounter the parameters (76,21,2,7) as in 3.3. They will appear one more time in Chapter 6.

4.3. The Non-Design Constructibility of $G = G(57,14,1,4)$

The existence of any strongly regular $G = G(57,14,1,4)$ is unknown.

(4.3.1) Theorem. There is no design constructible graph G for which $G = G(57,14,1,4)$.

Proof. Let G be any strongly regular graph having the stated parameters. By (3.2.3) we require a design D having parameters (15,42,14,5,4). Because $c = 1$, $\lambda = 4$, (4.2.4) and (4.2.6) indicate that $m_5 = 0$. Such a design exists (see Hall, 1967). We will show, however, that no design can afford G. Let $\beta = (o_1, o_2, o_3, o_4, o_5)$ denote an arbitrary block of D. $\Gamma_{G_B}(\beta)$ appears in Figure 4.

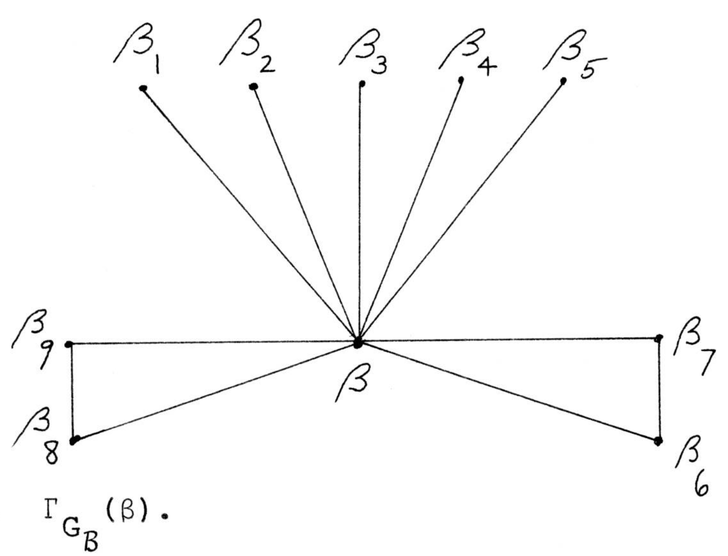

Figure 4. $\Gamma_{G_B}(\beta)$.

Here $\{\beta_i \cap \beta\} = \{o_i\}$, $1 \leq i \leq 5$, while $\{\beta \cap \beta_i\} = \phi$ if $6 \leq i \leq 9$. Also $\{\beta_6 \cap \beta_7\} = \{\beta_8 \cap \beta_9\} = \phi$. Since $v = 3k$, each of $\{\beta, \beta_6, \beta_7\}$ and $\{\beta, \beta_8, \beta_9\}$ constitute partitions of the fifteen objects. Hence each of $\{\beta_6 \cap \beta_8\}$ and $\{\beta_6 \cap \beta_9\}$ are nonempty, and similarly for $\{\beta_7 \cap \beta_8\}$ and $\{\beta_7 \cap \beta_9\}$. Furthermore $\{\beta_6 \cap \beta_8\} \cup \{\beta_6 \cap \beta_9\} = \beta_6$, $\{\beta_7 \cap \beta_8\} \cup \{\beta_7 \cap \beta_9\} = \beta_7$, $\{\beta_8 \cap \beta_6\} \cup \{\beta_8 \cap \beta_7\} = \beta_8$, $\{\beta_9 \cap \beta_6\} \cup \{\beta_9 \cap \beta_7\} = \beta_9$. Since $\lambda = 4$ we have $|\beta_i \cap \beta_j| \leq 3$, $i = 6, 7$; $j = 8, 9$. Thus, without loss of generality, $|\beta_6 \cap \beta_8| = 3$, $|\beta_6 \cap \beta_9| = 2$, $|\beta_7 \cap \beta_8| = 2$, and $|\beta_7 \cap \beta_9| = 3$.

Consider the subgraph G_{B_t} of G_B obtained by removing all edges having no triangles on them.

<u>Claim</u>. The situations pictured in Figure 5 are impossible in G_{B_t}.

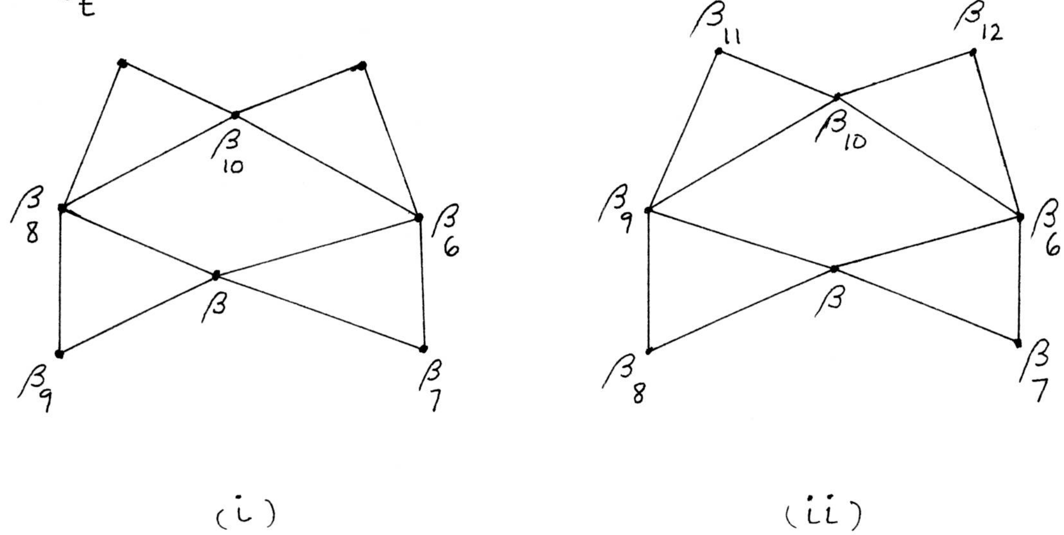

Figure 5. Situations in G_{B_t}.

<u>Proof of the Claim</u>. Figure 5(i) is impossible since $|\beta_6 \cap \beta_8| = 3$ and we would then have five paths of length two between β_6 and β_8 in G, violating $\lambda = 4$. As for Figure 5(ii), note that there are two paths of length two between β and β_{10} by virtue of β_9 and β_6. Now $|\beta \cap \beta_{10}| = 2$ or 3. Since $\lambda = 4$ we get $|\beta \cap \beta_{10}| = 2$ so that $|\beta_{10} \cap \beta_8| = |\beta_{10} \cap \beta_7| = 3$. Hence $\beta_{10} = (o_1, o_2, o_x, o_y, o_z)$, say, with x, y, $z \in \{6, 7, \ldots 15\}$. So $\{\beta_8 \cap \beta\} = \{\beta_7 \cap \beta\} = \phi$ requires $\{\beta_8 \cap \beta_{10}\} = \{o_x, o_y, o_z\} = \{\beta_7 \cap \beta_{10}\}$, and therefore $|\beta_8 \cap \beta_7| = 3$, a contradiction. We now have that every block β is associated with a configuration of the type in Figure 6.

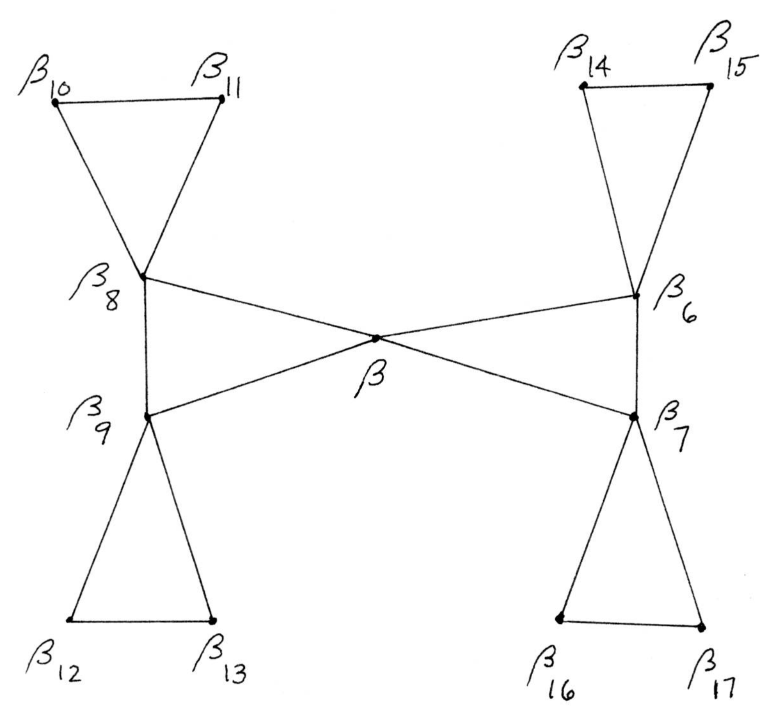

Figure 6. Blocks at Distance 1 or 2 from β in G_{B_t}.

All blocks pictured in Figure 6 are distinct.

Consider now the intersection distribution of β. We need $m_0 \geq 4$ because of β_6, β_7, β_8, β_9; $m_1 \geq 5$ because of β_1, β_2, ..., β_5, and finally, because of Figure 6 we need m_3, $m_2 \geq 4$. The Stanton-Sprott equations (4.2.2) say that

$$\text{(4.3.2)} \qquad \begin{cases} \sum_{i=0}^{4} m_i = 41 \\ \sum_{i=1}^{4} im_i = 65 \\ \sum_{i=1}^{4} i^2 m_i = 125 \end{cases} .$$

Recall that $m_5 = 0$ is required. It is routine to check that (4.3.2) admits no positive integer solution for which m_0, $m_1 - 1$, m_2, $m_3 \geq 4$. Hence no strongly regular $G = G(57,14,1,4)$ is d.c. □

CHAPTER 5

DESIGN CONSTRUCTIONS OF SOME KNOWN GRAPHS

There is an abundance of examples of known strongly regular graphs G that can be constructed by finding an appropriate design D affording G. Indeed, any strongly regular graph having a transitive automorphism group and possessing an object set is design constructible, the constancy of the block size being guaranteed by vertex transitivity. In the interest of symmetry, we present some of the more "coherent" constructions, that is, those for which the block adjacency rule is independent of block choice.

5.1. The Hoffman-Singleton Graph

Repeated reference has been made to the Hoffman-Singleton graph $G = G(50,7,0,1)$. We now establish its design constructibility.

(5.1.1) Theorem. The Hoffman-Singleton graph is afforded by PG(3,2).

Proof. From (3.1.2) we need a design $D = D(15,35,7,3,1)$. There are eighty nonisomorphic designs with these parameters (see Cole, White, and Cummings, 1925). We will first establish the reasons for singling out PG(3,2). For a block β, $\Gamma_{G_{\mathcal{B}}}(\beta)$ must consist of four mutually nonintersecting blocks $(\lambda = 1)$, which are all disjoint from β $(c = 0)$. Writing $\Gamma_{G_{\mathcal{B}}}(\beta) = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ we require then that $\{\beta, \beta_1, \beta_2, \beta_3, \beta_4\}$ constitute a complete replication of the fifteen objects, i.e., $\{\beta \cup \Gamma_{G_{\mathcal{B}}}(\beta)\}$ is a parallel class. Having chosen $\Gamma_{G_{\mathcal{B}}}(\beta)$, we then look at β_i $(i = 1,2,3,4)$. Each $\{\beta_i \cup \Gamma_{G_{\mathcal{B}}}(\beta_i)\}$, $1 \leq i \leq 4$, must be a parallel class, and since $\lambda = 1$, $c = 0$ we have $\Gamma_{G_{\mathcal{B}}}(\beta_i) \cap \Gamma_{G_{\mathcal{B}}}(\beta_j) = \beta$ $1 \leq i,\ j \leq 4,\ i \neq j$, (see Figure 7).

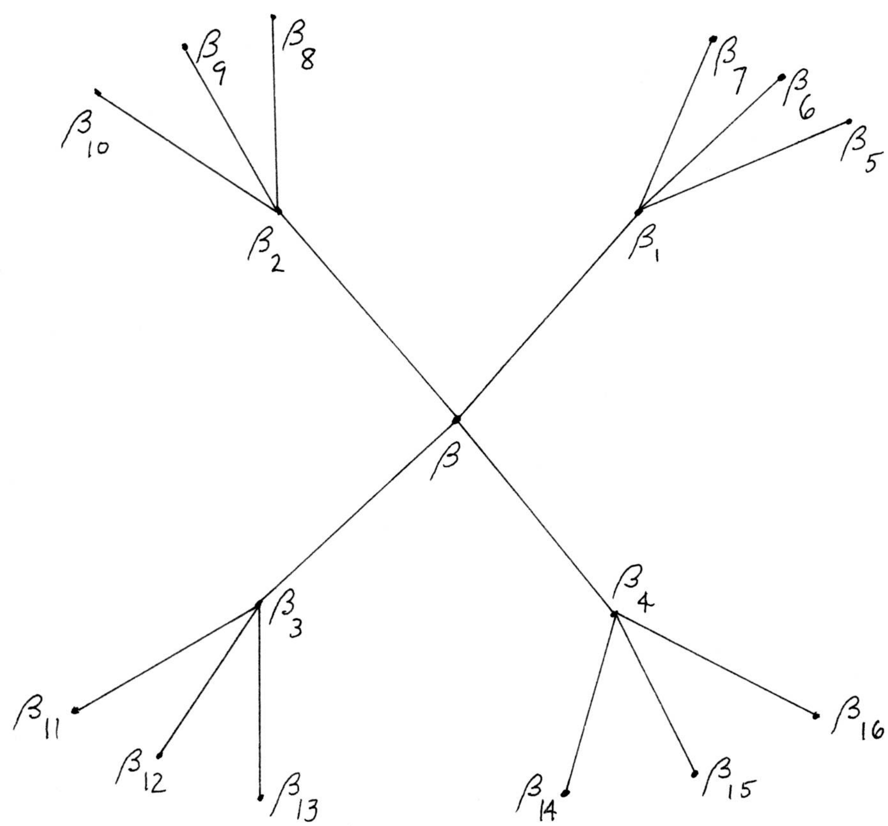

Figure 7. $\Gamma_{G_{\mathcal{B}}}(\beta)$, $\Gamma_{G_{\mathcal{B}}}(\beta_i)$ $(1 \leq i \leq 4)$.

Consider the seven blocks $\{\beta_1^j, \beta_2^j, \ldots, \beta_7^j\}$ containing the common object j, $j = 1, 2, \ldots, 15$. Since $\lambda = 1$, $\Gamma_{G_{\mathcal{B}}}(\beta_i^j) \cap \Gamma_{G_{\mathcal{B}}}(\beta_\ell^j) = \emptyset$ for $i, \ell = 1, \ldots, 7$, $i \neq \ell$. Thus the design must be resolvable, i.e., we must be able to partition its thirty-five blocks into seven parallel classes.

The resolvability requirement is precisely what led to the Kirkman Schoolgirl problem, posed and solved by Kirkman (1847). A schoolteacher takes her class of fifteen girls on a daily walk. The girls are arranged in five rows of three each, so that each girl has two companions. The problem is to arrange the girls so that for seven consecutive days no girl walks with one of her companions in a triplet more than once.

The Kirkman design thus obtained is isomorphic to the design obtained by taking as objects the fifteen points of $PG(3,2)$ and as blocks the thirty-five lines of $PG(3,2)$. Not only is $PG(3,2)$ resolvable but it has the largest automorphism group $(PSL(4,2) \cong A_8)$ among the eighty designs with the same parameters (see cole et al., 1925).

The following resolution of $PG(3,2)$ serves to give a new construction for the Hoffman-Singleton graph.

(5.1.2) PG(3,2) Resolved

Parallel Class	
1	{(1,2,3),(6,8,14),(5,9,12),(4,11,15),(7,10,13)}
2	{(1,4,5),(2,8,10),(6,9,15),(3,13,14),(7,11,12)}
3	{(1,6,7),(2,9,11),(5,8,13),(4,10,14),(3,12,15)}
4	{(1,8,9),(3,4,7),(2,13,15),(6,10,12),(5,11,14)}
5	{(1,10,11),(2,12,14),(3,5,6),(4,9,13),(7,8,15)}
6	{(1,12,13),(2,4,6),(3,8,11),(5,10,15),(7,9,14)}
7	{(1,14,15),(2,5,7),(4,8,12),(6,11,13),(3,9,10)}.

The geometry PG(3,2) has another property that can be exploited, viz., given two disjoint blocks β, $\bar{\beta}$, there exist exactly two parallel classes containing β and $\bar{\beta}$; and these classes have only β, $\bar{\beta}$ in common. For example, with (1,2,3) and (5,9,12) we get {(4,11,15),(6,8,14),(7,10,13)} and {(4,10,14),(7,8,15),(6,11,13)}.

We now present the rule describing block adjacency in $G_{\mathcal{B}}$.

(5.1.3) (i) Any block of the form (1,i,j) is connected to the four blocks in the same parallel class (in (5.1.2)) as (1,i,j).

(ii) Any block $\bar{\beta}$ of the form $\bar{\beta} = (i,j,k)$, $i, j, k \neq 1$, is connected to the block β which contains 1 and is in the same parallel class (in (5.1.2)) as $\bar{\beta}$, and to the uniquely determined three further blocks which form another parallel class on β and $\bar{\beta}$.

Thus for example we have the situation of Figure 8 for the blocks at distances one and two from (1,2,3) in G_B.

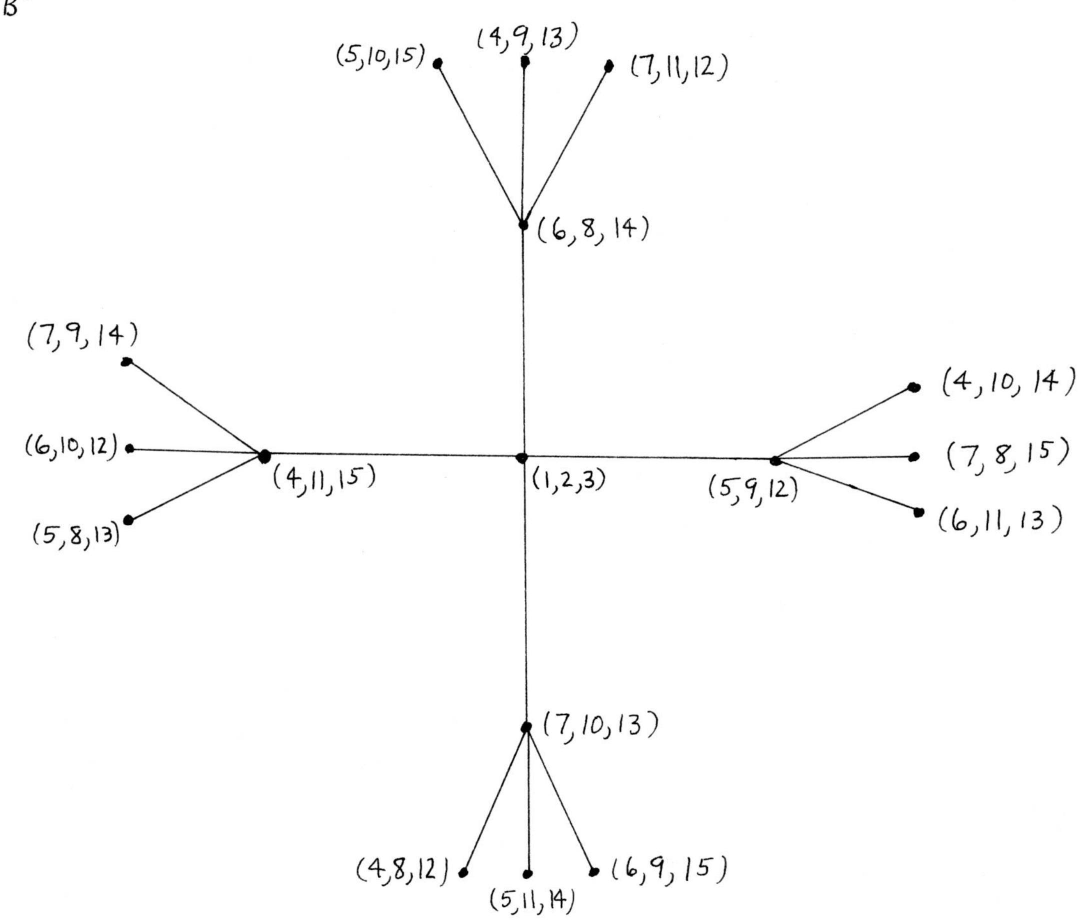

Figure 8. $\{\beta \in PG(3,2): \partial_{G_B}(\beta,(1,2,3)) \leq 2\}$.

It is routine to check that (5.1.3) defines the correct block adjacency rule, thus providing a design construction for the Hoffman-Singleton graph. (For the full $G_{\mathcal{B}}$ see Figure 10.) □

A design affording the next Moore graph $G = G(3250,57,0,1)$ would have to have parameters $(400,2850,57,8,1)$, which are precisely the parameters of PG(3,7). We conjecture that a resolution approach similar to the one employed in (5.1.1) should be taken. The difficulty here, of course, is the size of the design.

5.2. The Lattice Graphs and Their Complements

We have already provided design construction for an infinite family of strongly regular graphs in 3.2, namely the triangle graphs $T(m)$. We consider next another infinite family and describe a design construction for each graph as well as its complement.

The lattice graphs are strongly regular graphs $L(m)$, $m \geq 2$, having $n = m^2$, $r = 2(m - 1)$, $c = m - 2$, and $\lambda = 2$. The complementary lattice graphs $L'(m)$ have parameters $n' = m^2$, $r' = (m - 1)^2$, $c' = (m - 2)^2$, $\lambda' = (m - 1)(m - 2)$ (see Cameron and Van Lint, 1975).

(5.2.1) Theorem. The lattice graphs $L(m)$ and their complements $L'(m)$ are design constructible. In fact,

(i) $L(m)$ is afforded by two copies of K_2^m, and

(ii) $L'(m)$ is afforded by $m - 1$ copies of K_{m-1}^m.

<u>Proof</u>. For (i), we use (3.2.3) to discern that $L(m)$ requires a design $D(m)$ with parameters $(m,2\binom{m}{2},2(m-1),2,2)$. Taking $D(m)$ to be two copies of K_2^m, represent the objects as $\{1,2, \ldots, m\}$ and the blocks of $D(m)$ as $\{(1,2)^0,(2,1)^1,(1,3)^0,(3,1)^1, \ldots, (m-1,m)^0,(m,m-1)^1\}$ where the superscript indicates the respective copy and is taken modulo 2. We write $(i,j)^{\ell}$ with $i < j$ if $\ell = 0$, $i > j$ if $\ell = 1$. We take the following block adjacency rule:

$$\Gamma_{G_{\mathcal{B}}}((i,j)^0) = \{(i,\ell)^0 : \ell \neq j\} \cup \{(p,j)^1 : j > p\},$$

$$\Gamma_{G_{\mathcal{B}}}((i,j)^1) = \{(\ell,j)^1 : \ell \neq i\} \cup \{(i,p)^0 : i < p\}.$$

It is routine to check that this establishes (5.2.1)(i). For example, $L(4)$ is shown in Figure 9. For $L'(m)$, we use (3.2.3) to see that an affording design $D'(m)$ must have parameters $(m,2\binom{m}{2},(m-1)^2,m-1,(m-1)(m-2))$ which we may take to be $m - 1$ copies of K_{m-1}^m (this is the family referred to in 4.2, having $\alpha = 1$). Represent the $n - 1$ identical copies of K_{m-1}^m as D_1, $D_2, \ldots, D_{m-1}$. The block adjacency rule for a block β^i in

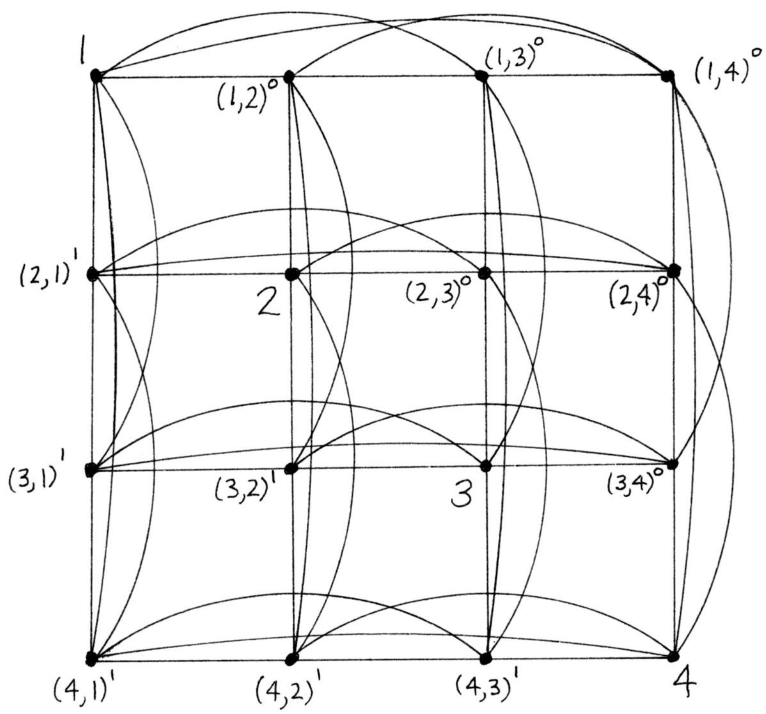

Figure 9. L(4).

D_i $(1 \leq i \leq m - 1)$ is $\Gamma_{G_{\mathcal{B}}}(\beta^i) = \{\beta \in D_j : j \neq i, \beta \neq \beta^i\}$. Again it is routine to verify the design construction and hence establish (5.2.1)(ii). □

5.3. Quasi-Symmetric Multiple Graphs

As was mentioned in Chapter 4, quasi-symmetric designs afford strongly regular graphs when the blocks comprise all of the vertices. Here two blocks are connected if and only if they intersect in a certain order. This is not the only use that can be made of such designs for constructing strongly regular graphs. We present here two known graphs, each constructed in a new way via quasi-symmetric designs.

A well known family of strongly regular graphs is obtained from the projective geometries PG(3,q), wherein the vertices of the graph are the $n = (q^2 + 1)(q^2 + q + 1)$ lines of the geometry and two lines are adjacent if and only if they intersect (see Biggs, 1971). The parameters of the resulting graph are

$$((q^2 + 1)(q^2 + q + 1), q(q + 1)^2, 2q^2 + q - 1, (q + 1)^2).$$

We present the design construction of PG(3,2).

<u>(5.3.1) Theorem</u>. The graph $G = G(35,18,9,9)$ is afforded by three copies D_1, D_2, D_3 of K_3^5 by means of the following rule: if a block of copy μ is denoted by β^μ then

$$\Gamma_{G_{\mathcal{B}}}(\beta^\mu) = \{\bar{\beta}^\mu \in D^\mu : |\beta^\mu \cap \bar{\beta}^\mu| = 1\} \cup$$

$$\{\bar{\beta}^\sigma \in D^\sigma : \sigma \neq \mu \ |\bar{\beta}^\sigma \cap \beta^\mu| = 2\}.$$

<u>Proof</u>. To afford G(35,18,9,9) we require a design D with parameters (5,30,18,3,9), which we may take to be three identical copies of K_3^5. Observe that K_3^5 has intersection distribution {0,3,6,0} for each block

(quasi-symmetric), so that the rule is well defined. Let B denote the 5×10 object-block incidence matrix of K_3^5 and let $Y = B^T B$. Let S be the 10×10 matrix that is 1 wherever Y is 1 and 0 otherwise. Then

$$(5.3.2) \qquad Y = 3I + S + 2(J - S - I) = I - S + 2J.$$

The block adjacency rule states that C, the block adjacency matrix, is given by

$$(5.3.3) \qquad C = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \end{array} \begin{array}{c} \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\ \begin{pmatrix} S & J - S - I & J - S - I \\ J - S - I & S & J - S - I \\ J - S - I & J - S - I & S \end{pmatrix} \end{array}.$$

By (4.1.13) and its proof it suffices to show that

$$(5.3.4) \qquad \begin{cases} \text{(i)} & [B:B:B]C = 9J_{5\times 30}, \\ \text{(ii)} & C^2 = 9I + 9J - \begin{pmatrix} Y & Y & Y \\ Y & Y & Y \\ Y & Y & Y \end{pmatrix}. \end{cases}$$

Note first that

$$BJ_{5\times 10} = 6J_{5\times 10}, \qquad J_{5\times 5}B = 3J_{5\times 5}.$$

Hence

$$BS = B(I + 2J - Y) = B + 12J_{5\times 10} - BB^{T}B.$$

Using (2.4.3) we have

$$BS = B + 12J_{5\times 10} - (3I + 3J_{5\times 5})B,$$

or

$$BS = B + 12J_{5\times 10} - 3B - 9J_{5\times 10} = 3J_{5\times 10} - 2B.$$

Thus

$$[B:B:B]C$$

$$= [-BS + 2BJ_{10\times 10} - 2B : -BS + 2BJ_{10\times 10} - 2B : -BS + 2BJ_{10\times 10} - 2B]$$

or

$$[B:B:B]C = [9J_{5\times 10} : 9J_{5\times 10} : 9J_{5\times 10}],$$

establishing (5.3.4)(i).

As for (5.3.4)(ii) we note that

$$(5.3.5) \qquad SJ = JS = 3J$$

while

$$(S^2)_{ij} = \begin{cases} 3 & \text{if } i = j, \\ 1 & \text{if } S_{ij} = 0, \quad i \neq j, \\ 0 & \text{if } S_{ij} = 1, \end{cases}$$

so

$$(5.3.6) \qquad S^2 = J + 2I - S.$$

Setting $T = J - S - I$ we have

$$C^2 = \begin{pmatrix} S^2 + 2T^2 & 2ST + T^2 & 2ST + T^2 \\ 2ST + T^2 & S^2 + 2T^2 & 2ST + T^2 \\ 2ST + T^2 & 2ST + T^2 & S^2 + 2T^2 \end{pmatrix},$$

which reduces via (5.3.6), (5.3.5), and (5.3.2) to

$$C^2 = \begin{pmatrix} 9I & & 0 \\ 0 & 9I & 0 \\ 0 & 0 & 9I \end{pmatrix} + 9\begin{pmatrix} J & J & J \\ J & J & J \\ J & J & J \end{pmatrix} - \begin{pmatrix} Y & Y & Y \\ Y & Y & Y \\ Y & Y & Y \end{pmatrix}$$

so that (5.3.4)(ii) is in force. Hence the construction. □

We present one final construction using similar techniques. It is included to introduce a design that will afford two graphs in the present situation and will afford yet another construction in Chapter 7.

A (design-free) construction of a graph $G = G(36,15,6,6)$ is cited in Biggs (1971). We prove:

(5.3.7) Theorem. The graph $G = G(36,15,6,6)$ is afforded by three copies $\bar{D}_1$, $\bar{D}_2$, $\bar{D}_3$ of the unique design $\bar{D} = \bar{D}(6,10,5,3,2)$ (see (4.1.15)) by means of the rule: if β^μ is a block of D_μ then

$$\Gamma_{G_{\mathcal{B}}}(\beta^\mu) = \{\bar{\beta}^\mu \in D^\mu : |\bar{\beta}^\mu \cap \beta^\mu| = 1\} \cup$$

$$\{\bar{\beta}^\sigma \in D^\sigma : \mu \neq \sigma \ |\bar{\beta}^\sigma \cap \beta^\mu| = 2\}.$$

Proof. A design with parameters $(6,30,15,3,6)$ is required; note that $\bar{D}$ has intersection distribution $\{0,6,3,0\}$ for each block, so $\bar{D}$ is quasi-symmetric. Carry out the exact same procedure as in the proof of (5.3.1). □

We note that both K_3^5 and $\bar{D}$ give rise to a familiar strongly regular graph.

(5.3.3) Corollary. There are three disjoint copies of the Petersen graph in the graph $G(35,18,9,9)$ constructed in (5.3.1), while the graph $G(36,15,6,6)$ constructed in (5.3.7) contains three disjoint copies of the complement of the Petersen graph.

Proof. In the proof for $G(35,18,9,9)$, S satisfied (5.3.5) and (5.3.6), making it (see (2.3.4) and (2.3.5)) the adjacency matrix of a strongly regular graph with parameters $(10,3,0,1)$. The three Petersen graphs arise from the three copies of S used in (5.3.3). For $G(36,15,6,6)$ the 10×10 matrix $\bar{S}$ whose rows and columns are indexed by the blocks of $\bar{D}$ and for which

$$\bar{s}_{ij} = \begin{cases} 1 & \text{if } |\beta_i \cap \beta_j| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

is seen to satisfy $\bar{S}^2 = 2I - \bar{S} + 4J$ and $\bar{S}J = 6J$. Thus $\bar{S}$ is (via (2.3.4) and (2.3.5)) the adjacency matrix of a strongly regular graph whose parameters are $(10,6,3,4)$, the complement of the Petersen graph. The three copies of this graph arise from the three copies of $\bar{D}$ used in (5.3.7). □

CHAPTER 6

THE BLOCK GRAPH

The purpose of this chapter is to view the block graph as a coherent configuration of minimal rank. We will consider those cases for which the configuration is homogeneous of rank 3 and trivially paired, i.e., strongly regular (in 6.1), or of rank 4 and distance regular (in 6.2), or of larger rank but with the adjacency relations defined solely in terms of graph distance and block intersection order (in 6.3).

6.1. Block Graphs Having Fewer Than Four Distinct Eigenvalues

Our main source of information about the pair $G(G(n,r,c,\lambda),\ D = D(v,b,r,k,\lambda)$ is the block graph spectrum. Recall that

$$(6.1.1)\qquad \mathrm{Spec}(C) = \begin{pmatrix} r-k & -k & s-k & s-2k \\ 1 & z_2 - v & z_3 - (v-1) & v-1 \end{pmatrix}.$$

We consider block graphs that have a degenerate spectrum in the sense that there are three or fewer distinct eigenvalues.

<u>(6.1.2) Theorem</u>. Let $G_{\mathcal{B}}$ have fewer than four distinct eigenvalues, then one of the following holds:

(i) $G = K_{r,r}$

(ii) $G = G(10,3,0,1)$, the Petersen graph, or

(iii) $G_{\mathcal{B}}$ is strongly regular.

<u>Proof</u>. We first treat the situation in which C has but one (distinct) eigenvalue. In this case $G_{\mathcal{B}}$ is a void graph (having no edges) so that $r - k = -k = s - k = s - 2k = 0$, making $r = 0$, a contradiction to our assumptions in (2.3.9). If C has just two distinct eigenvalues then clearly $G_{\mathcal{B}} = K_b$ so that $r - k = b - 1$ and $-k = s - k = s - 2k = -1$, also impossible. If C has just three distinct eigenvalues then either some multiplicity is 0 or else two of the eigenvalues listed in (6.1.1) are equal. In the latter case one of the following must occur:

(6.1.3)
(i) $r - k = -k$
(ii) $r - k = s - k$,
(iii) $r - k = s - 2k$,
(iv) $-k = s - k$,
(v) $-k = s - 2k$,
(vi) $s - k = s - 2k$.

Since r is assumed positive (6.1.3)(i) is eliminated.

For the situation of (6.1.3)(iii) we use $k^2 - k(\lambda - c) + \lambda - r = 0$ (from (3.2.5)) to get $rk = r - \lambda$ (recall that $\lambda - c = 2k - s$ in (3.3.6)). Hence by $\lambda(v - 1) = r(k - 1)$ in (2.4.2), we obtain $v = 0$, an impossibility.

For (6.1.3)(ii) we have $r = s$ or, since $s = \sqrt{(\lambda-c)^2+4(r-\lambda)}$, we have $\lambda^2 - \lambda(2c + 4) + c^2 + 4r - r^2 = 0$. Hence $(r - 2)^2 + 4c$ is a square. Since $c < r - 1$, in (2.3.9), we have that $(r - 2)^2 + 4c$ is a square if and only if $c = 0$.

For $c = 0$ we have $r = \lambda$ or $r = 4 - \lambda$ according as $\lambda \geq 2$ or $\lambda = 1$. If $r = \lambda$ then $G = K_{r,r}$. If $c = 0$, $\lambda = 1$, $r = 4 - \lambda = 3$ then $G = G(10,3,0,1)$.

In case one of (6.1.3)(iv), (v), or (vi) holds, or if some multiplicity in C is 0, we have (by the Perron-Frobenius Theorem) a connected block graph with three distinct eigenvalues. We shall prove:

<u>(6.1.4) Proposition</u>. If the adjacency matrix C of a regular connected graph $G_{\mathcal{B}}$ has only three distinct eigenvalues then $G_{\mathcal{B}}$ is strongly regular.

<u>Proof</u>. Let $\mathrm{spec}(C) = \begin{pmatrix} \rho & \rho_2 & \rho_3 \\ 1 & y_2 & y_3 \end{pmatrix}$, where ρ is the valence of $G_{\mathcal{B}}$. Since C has a cubic minimal polynomial $\{I, C, C^2\}$ generates a 3-dimensional R-algebra $A(C)$ which is commutative and consists entirely of symmetric matrices.

Write $\vec{j} = (1,1, \ldots, 1)$. Since $G_{\mathcal{B}}$ is regular and connected $\vec{j}$ spans the eigenspace corresponding to the eigenvalue ρ (by the Perron-Frobenius Theorem). Writing the minimum polynomial of C as $(\lambda - \rho)q(\lambda)$ whe have $Cq(C) = \rho q(C)$, i.e., each column of $q(C)$ is an eigenvector for ρ. Hence $q(C) = \alpha J$ for some $\alpha \in R$. Thus $J \in A(C)$ and hence also $J - C - I \in A(C)$. Thus $\{I,J,J - C - I\}$ is a basis for $A(C)$, and there exist nonnegative integers c, λ such that $C^2 = \rho I + cC + \lambda(J - C - I)$. According to (2.3.4) and (2.3.5), $G_{\mathcal{B}}$ is strongly regular, proving the proposition and completing the proof of the theorem. □

We are therefore interested in the situations for which $G_{\mathcal{B}}$ is connected and C has but three distinct eigenvalues, that is G is d.c. and $G_{\mathcal{B}}$ is strongly regular.

<u>(6.1.5) Definition</u>. If $G_1, G_2, \ldots, G_m$ are all strongly regular and $G_2, G_3, \ldots, G_m$ are d.c., with $G_i = (G_{i+1})_{\mathcal{B}}$ for $1 \leq i \leq m - 1$ we will write $G_1 \triangleleft_{\mathcal{B}} G_2 \triangleleft_{\mathcal{B}} G_3 \ldots \triangleleft_{\mathcal{B}} G_m$ and say that we have a <u>chain</u> of length m.

In the terminology of 3.3 the graphs G_1, $G_2, \ldots, G_{m-1}$ are strongly regular type $(-k)$ eigengraphs. In an effort to classify the phenomenon of (3.3.8)(ii) we establish the next theorem.

<u>(6.1.6) Theorem</u>. If $H \triangleleft_{\mathcal{B}} G(n,r,c,\lambda)$ is a chain of length two then either

(i) $G = K_{k,k,\ldots,k}$, the complete $(\frac{r}{k}+1)$ - partite graph,

or

(ii) $z_2 = v$.

<u>Proof</u>. Because of (6.1.4) we consider those situations for which C has three distinct eigenvalues. By (6.1.3) one of the following must occur:

(a) $-k = s - 2k$,

(b) $-k = s - k$,

(c) $s - k = s - 2k$,

(d) $z_2 - v = 0$,

(e) $z_3 - (v - 1) = 0$.

(We may rule out the case $v = 1$ since it occurs if and only if $G = K_n$).

In (a) we have $s = k$ since $c - \lambda = s - 2k$, hence (by (3.3.6)) $k = \sqrt{k^2+4(r-\lambda)}$, forcing $r = \lambda$. We are back to (4.2.9)(i) and $G = \underbrace{K_{k,k,\ldots k}}_{(\frac{r}{k}+1)}$, $H = \underbrace{K_{k,k,\ldots,k}}_{(\frac{r}{k})}$.

In (b) we have $s = 0$, forcing $\sqrt{(\lambda-c)^2+4(r-\lambda)} = 0$ or $r = c = \lambda$, a contradiction to the assumption $c < r - 1$ of (2.3.9).

In (c) we obtain $k = 0$ and $\lambda = c = r$, which is impossible.

In (e), we have by (2.3.8) that $((n-1)(-k)+r)/-s = v-1$ so that via (3.2.3)(ii) we obtain $s = r+k$. But this entails $s-k > s-2k > r-k$ a contradiction to the Perron-Frobenius Theorem. Hence only (6.1.6)(i) and (ii) survive and the theorem is proved. □

Henceforth we consider only chains that occur via (6.1.6)(ii). We need the parameter relationship between graphs in a chain.

<u>(6.1.7) Lemma</u>. Let $G_1 = G_1(n_1, r_1, c_1, \lambda_1)$ and $G_2 = G_2(n_2, r_2, c_2, \lambda_2)$ both be strongly regular. Let $G_1 \lhd\!\!\!\mathcal{B}\ G_2$ be a chain of length two (of type (6.1.6)(ii)), G_2 being afforded by $D_2 = D_2(v_2, b_2, r_2, k_2, \lambda_2)$ and let C_2 be the adjacency matrix of $(G_2)_{\mathcal{B}} = G_1$. Set $s_i = \sqrt{(\lambda_i - c_i)^2 + 4(r_i - \lambda_i)}$ $i = 1, 2$; then

$$\begin{cases} n_1 = b_2, \\ r_1 = r_2 - k_2, \\ c_1 = \lambda_1 - (3k_2 - 2s_2), \\ \lambda_1 = (s_2 - 2k_2)(s_2 - k_2) + r_1. \end{cases}$$

<u>Proof</u>. We have $z_2 = v_2$ so that (6.1.1) becomes

$$\operatorname{spec}(C_2) = \begin{Bmatrix} r_2 - k_2 & s_2 - k_2 & s_2 - 2k_2 \\ \\ 1 & b_2 - v_2 & v_2 - 1 \end{Bmatrix}.$$

Thus $n_1 = b_2$, $r_1 = r_2 - k_4$. Since $s_2 - 2k_2 < s_2 - k_2$, $s_2 - 2k_2$ is the unique negative eigenvalue of C_2, hence we write $-k_1 = s_2 - 2k_2$, $s_1 - k_1 = s_2 - k_2$ and

$$\operatorname{spec}(C_2) = \begin{Bmatrix} r_1 & -k_1 & s_1 - k_1 \\ \\ 1 & v_2 - 1 & b_2 - v_2 \end{Bmatrix}.$$

Now $s_1 = k_2$ and, since $k_1 = \frac{1}{2}(\lambda_1 - c_1 + s_1)$, $\lambda_1 - c_1 = 3k_2 - 2s_2$. From

$$s_1^2 = (\lambda_1 - c_1)^2 + 4(r_1 - \lambda_1)$$

we have

$$k_2^2 = (3k_2 - 2s_2)^2 + 4(r_1 - \lambda_1),$$

so that

$$\lambda_1 = (2k_2 - s_2)(k_2 - s_2) + r_1. \qquad \square$$

The next theorem will enable us to classify all instances of $G_1 \triangleleft_{\mathcal{B}} G_2$ of type (6.1.6)(ii).

<u>(6.1.8) Theorem.</u> In case $G_1 \triangleleft_{\mathcal{B}} G_2$ as in (6.1.6)(ii) with D_2 affording G_2 then $\mathrm{Aut}(D_2) = \mathrm{Aut}(G_1)$.

<u>Proof.</u> Certainly $\mathrm{Aut}(G_1) = \mathrm{Aut}((G_2)_{\mathcal{B}}) < \mathrm{Aut}(D_2)$. Let P be a permutation matrix representing a permutation of the objects of D_2 which preserves the blocks of D_2. Hence P induces a permutation matrix Q of the blocks for which $PB_2Q = B_2$, where B_2 is the object-block incidence matrix of D_2. Write $Y_2 = B_2^T B_2$ so that $Y_2 = Q^T Y_2 Q$. We will show that $Q^T C_2 Q = C_2$, proving the theorem.

Recall, from (4.1.3)(i), that

$$C_2^2 = (r_2 - \lambda_2)I + (c_2 - \lambda_2)C_2 + \lambda_2 J - Y_2. \tag{6.1.9}$$

Since C_2 is the adjacency matrix of the strongly regular graph G_1 we also have, by (2.3.5)

$$C_2^2 = (r_1 - \lambda_1)I + (c_1 - \lambda_1)C_2 + \lambda_1 J. \tag{6.1.10}$$

Combining (6.1.9) with (6.1.10) and using (6.1.7) we obtain

$$Y_2 = \mu_1 I + \mu_2 C_2 + \mu_3 J, \tag{6.1.11}$$

where

$$\mu_1 = r_2 - \lambda_2 + (2k_2 - s_2)(k_2 - s_2)$$

$$\mu_2 = k_2 - s_2$$

$$\mu_3 = \lambda_2 - (2k_2 - s_2)(k_2 - s_2) + k_2 - r_2.$$

In case $\mu_2 = 0$ we have situation (6.1.6)(i). Otherwise, by left multiplying by Q^T and right multiplying by Q in (6.1.11) we see that $Q^T C_2 Q = C_2$, and the theorem is proved. □

<u>(6.1.12) Theorem</u>. In order for $G_2 = G_2(n_2, r_2, c_2, \lambda_2)$ to be afforded by $D_2 = D_2(v_2, b_2, r_2, k_2, \lambda_2)$ and form a chain $G_1 \lhd_B G_2$ of length two (of type (6.1.6)(ii)) with $G_1 = G_1(n_1, r_1, c_1, \lambda_1)$ it is necessary and sufficient that D_2 be quasi-symmetric with the following parameters:

$$v_2 = \frac{k_2^2 + k_2\delta - \delta - \delta^2}{k_2 - \delta^2},$$

$$b_2 = \frac{(k_2+\delta)(k_2^2 + k_2\delta - \delta - \delta^2)}{(k_2 - \delta^2)(\delta+1)},$$

$$r_2 = \frac{k_2(k_2+\delta)}{\delta+1},$$

$$k_2 = k_2,$$

$$\lambda_2 = \frac{k_2(k_2-\delta^2)}{\delta+1},$$

where $\delta = s_2 - k_2$ and every block pair in D_2 intersects in $x = k_2 - \delta^2 - \delta$ or $y = x + \delta$ objects, two blocks of D_2 being adjacent in G_1 if and only if they intersect in x objects.

<u>Proof</u>. We first establish necessity. From (6.1.11) we see that D_2 must be quasi-symmetric and that two blocks are adjacent if and only if they intersect in $\mu_2 + \mu_3$ objects. The intersection orders of D_2 are thus $\mu_2 + \mu_3$ and μ_3. Note that $\mu_2 + \mu_3 < \mu_3$ since $\mu_2 = -\delta < 0$ (as in the proof of (6.1.7)). Denote $x = \mu_2 + \mu_3$, $y = \mu_3$ then by $k_2^2 - (\lambda_2 - c_2)k_2 - (r_2 - \lambda_2) = 0$ of (3.2.5) we have $r_2 - \lambda_2 = k_2(\delta)$, so that

(6.1.13) $$x = k_2 - \delta - \delta^2, \qquad y = x + \delta.$$

By the block adjacency rule wherein blocks are adjacent if and only if their intersection order is x we have

(6.1.14) $$m_x = r_2 - k_2.$$

We now apply the Stanton-Sprott Equations (4.2.3) to D_2 to obtain

(6.1.15)
$$\begin{cases} m_x + m_y = b_2 - 1, \\ xm_x + ym_y = k_2(r_2 - 1) \\ x^2m_x + y^2m_y = k_2(r_2 - 1) + k_2(k_2 - 1)(\lambda_2 - 1). \end{cases}$$

Eliminate m_y from (6.1.15), solve for m_x and combine with (6.1.14) to yield

(6.1.16) $$r_2 - k_2 = \frac{(k_2-\delta^2)(b_2-1)-k_2(r_2-1)}{\delta}.$$

Solving (6.1.16) for b_2 we have

(6.1.17) $$b_2 = \frac{(\delta+k_2)(r_2-\delta)}{k_2-\delta^2}.$$

From $\lambda_2 - c_2 = 2k_2 - s_2$ (in (3.3.6)), $r_2 - \lambda_2 = k_2\delta$, (6.1.13), and Lemma (6.1.7) we can write x, y, c_i, λ_i, $i = 1, 2$, all in terms of k_2 and δ as

$$(6.1.18)\quad \begin{cases} x = k_2 - \delta - \delta^2, \quad y = x + \delta, \\ c_2 = x + c_1, \\ \lambda_2 = r_2 - \delta k_2, \\ c_1 = \lambda_1 + 2\delta - k_2, \\ \lambda_1 = r_1 + \delta^2 - k_2\delta. \end{cases}$$

Distinguish an object o_1, of D_2 and consider $\Gamma_{G_2}(o_1)$. Say $\beta_1, \ldots, \beta_{r_2}$ each contain o_1. Say that $\beta_1 = (o_1, o_2, \ldots, o_k)$ and count in two ways the number of object pairs in D_2 of the form $\{o_1, o_j\}$ where $o_j \in \beta_1$. We have $\lambda_2(k_2 - 1)$ such pairs; on the other hand $k_2 - 1$ of the pairs appear in β_1 itself while $c_2(x - 1)$ appear in the blocks of $\Gamma(o_1)$ which are adjacent to β_1. The remaining pairs, $(r_2 - c_2 - 1)(y - 1)$ of them, lie in blocks that are in $\Gamma(o_1)$ but are not adjacent to β_1. Thus

$$(6.1.19)\quad \lambda_2(k_2 - 1) = k_2 - 1 + c_2(x - 1) + (r_2 - c_2 - 1)(y - 1).$$

Applying (6.1.18) to (6.1.19) gives us

$$(6.1.20)\quad c_2 = ((k_2 - 1)(\lambda_2 - 1) - (y - 1)(r_2 - 1))/-\delta = x + c_1.$$

Hence, from (6.1.18) with respect to c_1 we see by (6.1.20) that

$$\lambda_2(k_2 + \delta) = r_2(k_2 - \delta^2). \tag{6.1.21}$$

Now that r_2, λ_2, b_2 have been expressed in terms of k_2 and δ we can finish describing the parameters of D_2 by using the fact that $r_2(k_2 - 1) = \lambda_2(v_2 - 1)$ and complete the proof of necessity in the theorem.

For sufficiency, note that $G_1 \lhd_B B_2$ is a chain of length two of type (6.1.6)(ii) if and only if D_2 affords G_2 and $z_2 = v_2$. For the latter half we note from (2.3.8) that $z_2 = ((n_2 - 1)\delta + r_2)/s_2$. The requirement that $z_2 = v_2$ is then easily checked for the D_2 parameters given in the theorem. Hence, to finish, we prove that D_2 affords G_2.

We need only establish (4.1.3)(i) and (ii) as with (4.1.13). By the quasi-symmetry of D_2 we have

$$Y_2 = (k_2 - y)I + (x - y)C_2 + yJ \tag{6.1.22}$$

for the appropriate (0,1)-matrix C_2. Taking B_2 as the object-block incidence matrix of D_2 we use (6.1.22), (2.4.3), (2.4.7) and the stated D_2 parameters to obtain

(6.1.23) $$B_2C_2 = (x + 2\delta + \delta^2 - 2k_2)B_2 + \lambda_2 J.$$

Set

$$c_2 = \lambda_2 + x + 2\delta + \delta^2 - 2k_2.$$

Combining (6.1.22), (2.4.3), (2.4.7) and the stated D_2 parameters finally gives

(6.1.24) $$C_2^2 + Y_2 = (r_2 - \lambda_2)I + (c_2 - \lambda_2)C_2 + \lambda_2 J.$$

Hence we take C_2 to be the block adjacency matrix for G_2 and we have D_2 affording G_2. □

Shrikhande (1973) independently obtained results similar to (6.1.12). We now consider some examples.

<u>(6.1.25) Example</u>. Take $x = 0$, $\delta = 2$ in (6.1.12) and obtain, parametrically, $G_2 = G_2(77,16,0,4)$, $G_1 = G_1(56,10,0,2)$, $D_2 = D_2(21,56,16,6,4)$. The unique design D_2 with these parameters is known (see Cameron and Van Lint, 1975), and is quasi-symmetric with automorphism group PSL(3,4). Hence we can construct both G_1 and G_2 and use (6.1.8) to get $\mathrm{Aut}(G_1) = \mathrm{PSL}(3,4)$. Gewirtz (1969) has given alternate constructions of both G_1 and G_2 and has established their uniqueness. From the Gewirtz construction of G_1 it is easy to establish the design

constructibility of G_1. To afford G_1 a design must have parameters (16,40,10,4,2). Gewirtz's graph G_1 is afforded by the design D_1 whose blocks appear in (6.1.26). (To get the sixteen objects of D_1 from Gewirtz's construction, choose his ℓ_0 and any fifteen involutions in ℓ_2 wich fix a given number.) Note that D_1 has intersection distribution {9,24,6,0,0} having no blocks intersecting in three or four objects, as is consistent with the observations of 4.2. We believe D_1 to be a new design, the only other known design with these parameters being the design formed from two copies of EG(2,4).

(6.1.26) $D_1 = D_1(16,40,10,4,2)$

{(1,2,3,4),(1,2,5,6),(1,3,7,8),(1,5,9,10),(1,4,11,12),
(1,6,13,14),(1,7,13,15),(1,8,9,16),(1,10,11,15),(1,12,14,16),
(8,11,13,14),(7,9,10,12),(6,9,11,16),(5,12,13,15),(5,7,14,16),
(6,8,10,15),(9,11,12,13),(7,8,10,14),(3,10,12,6),
(3,11,14,15),(4,8,13,16),(4,7,9,15),(2,12,14,15),(5,6,15,16),
(2,10,11,16),(4,6,9,14),(4,5,10,13),(2,7,13,16),(3,4,15,16),
(4,6,8,12),(3,6,7,11),(2,8,9,15),(3,5,9,14),(3,6,10,13),
(4,5,7,11),(3,5,8,12),(2,4,10,14),(2,5,8,11),(2,6,7,12),
(2,3,9,13)}.

<u>(6.1.27) Example</u>. Take $x = 0$, $\delta = 3$ in (6.1.12) and get $G_2 = G_2(210,33,0,6)$, $G_1 = G_1(266,45,0,9)$,

$D_2 = D_2(56,210,45,12,9)$. If there exists such a quasi-symmetric design then we get constructions for G_1 and G_2, both listed as unknown by Gewirtz (1969).

One final example brings into play two graphs we have already mentioned.

<u>(6.1.28) Example</u>. With $x = 1$, $\delta = 2$ in (6.1.12) we have $G_2 = G_2(76,21,2,7)$, $G_1 = G_1(57,14,1,4)$, $D_2 = D_2(19,57,21,7,7)$. Sprott (1956) constructed a design having the parameters of D_2, but it is not quasi-symmetric, and to our knowledge no such quasi-symmetric design has been constructed. Recall that the elusive G_2 appeared in two other contexts in 3.3 and 4.2. We established the non-design constructibility of G_1 in (4.3.1).

In the positive direction we can use (6.1.12) to obtain an infinite chain.

<u>(6.1.29) Corollary</u>. The graphs G_1, G_2, G_3, ... with $G_i = T'(i + 4)$ (the triangle complement graphs described in (3.2.8)) form an infinite chain $G_1 \triangleleft\!\mathcal{B}] \; G_2 \triangleleft\!\mathcal{B}] \; G_3 \ldots$, and have $D_i = K_{i+1}^{i+3}$, $\mathrm{Aut}(G_i) = S_{i+4}$. In particular G_1 is the Petersen graph, with automorphism group S_5.

<u>Proof</u>. Take $\delta = 1$ and $k_2 = i + 1$, $i = 2, 3, \ldots$, in (6.1.12) and observe that K_{i+1}^{i+3} is quasi-symmetric with $x = i - 1$ and $y = i$. □

6.2. Distance Regular Block Graphs of Rank Four

Since $G_{\mathcal{B}}$ is regular and has at most four distinct eigenvalues $\operatorname{diam}(G_{\mathcal{B}}) \leq 3$ by a theorem of Biggs (1974). For use in this and the next section we begin with the following observation:

(6.2.1) Proposition. (i) Two blocks in $G_{\mathcal{B}}$ are at distance three if and only if they intersect in λ objects and are not adjacent in G.

(ii) In particular if $\lambda > c$ and $m_\lambda > 0$ (for some block's intersection distribution) then $\operatorname{diam}(G_{\mathcal{B}}) = 3$.

(iii) Moreover, if $\lambda > k$ then $\operatorname{diam}(G_{\mathcal{B}}) = 2$.

Proof. For (i) observe that for β, β' such that $\partial_G(\beta,\beta') = 2$, $|\beta \cap \beta'| = \lambda$ we have λ objects connected to both β and β' in G. Hence $\partial_{G_{\mathcal{B}}}(\beta,\beta') = 3$ and conversely. For (ii) note that with β, β' having $|\beta \cap \beta'| = \lambda > c$ we must have $\partial_G(\beta,\beta') = 2$ since otherwise $\langle \beta,\beta' \rangle$ has too many triangles. By part (i) $\operatorname{diam}(G_{\mathcal{B}}) = 3$. Finally, if $\lambda > k$ then $|\beta \cap \beta'| = \lambda$ is impossible, so $\operatorname{diam}(G_{\mathcal{B}}) = 2$. □

We consider now those block graphs of design constructible strongly regular graphs that are distance regular and of diameter three. Much less is known, generally, about distance regular rank four graphs than about strongly regular graphs. Very few constructions are available, largely because the approach of finding automorphism groups with rank four representations has not been studied with nearly the intensity as in the case of rank three groups. To motivate our rank four block graph considerations we give an example in the next theorem.

<u>(6.2.2) Theorem</u>. There exists a distance regular block graph of rank four with intersection array $\{4,3,3;1,1,2\}$.

<u>Proof</u>. Take G to be the design constructed Hoffman-Singleton graph from (5.1.1). Then $G_{\mathcal{B}}$ is the graph in question. The adjacency relations $\{I,f_1,f_2,f_3\}$ defined by $(\beta_i,\beta_j) \in f_1$ if and only if $|\beta_i \cap \beta_j| = 0$, $\partial_G(\beta_i,\beta_j) = 1$, $(\beta_i,\beta_j) \in f_2$ if and only if $|\beta_i \cap \beta_j| = 0$, $\partial_G(\beta_i,\beta_j) = 2$, and $(\beta_i,\beta_j) \in f_3$ if and only if $|\beta_i \cap \beta_j| = 1$ serve to construct a rank four homogeneous trivially paired coherent configuration. Thus we have a rank four distance regular graph, and the stated intersection array is easily verified. The graph is pictured in Figure 10. □

For the graph of Theorem (6.2.2) and for future reference we make the following definition.

<u>(6.2.3) Definition</u>. A <u>perfect one-error correcting code</u> in a graph H is a subset T of $V(H)$ for which $V(H) = \dot{\bigcup}\{\Gamma_H(t):t \in T\}$. The vertices of T are called

Figure 10. The Block Graph of the Hoffman-Singleton Graph.

The connections between the eighteen vertices that are three away from (1,2,3) are given by the three hexagons.

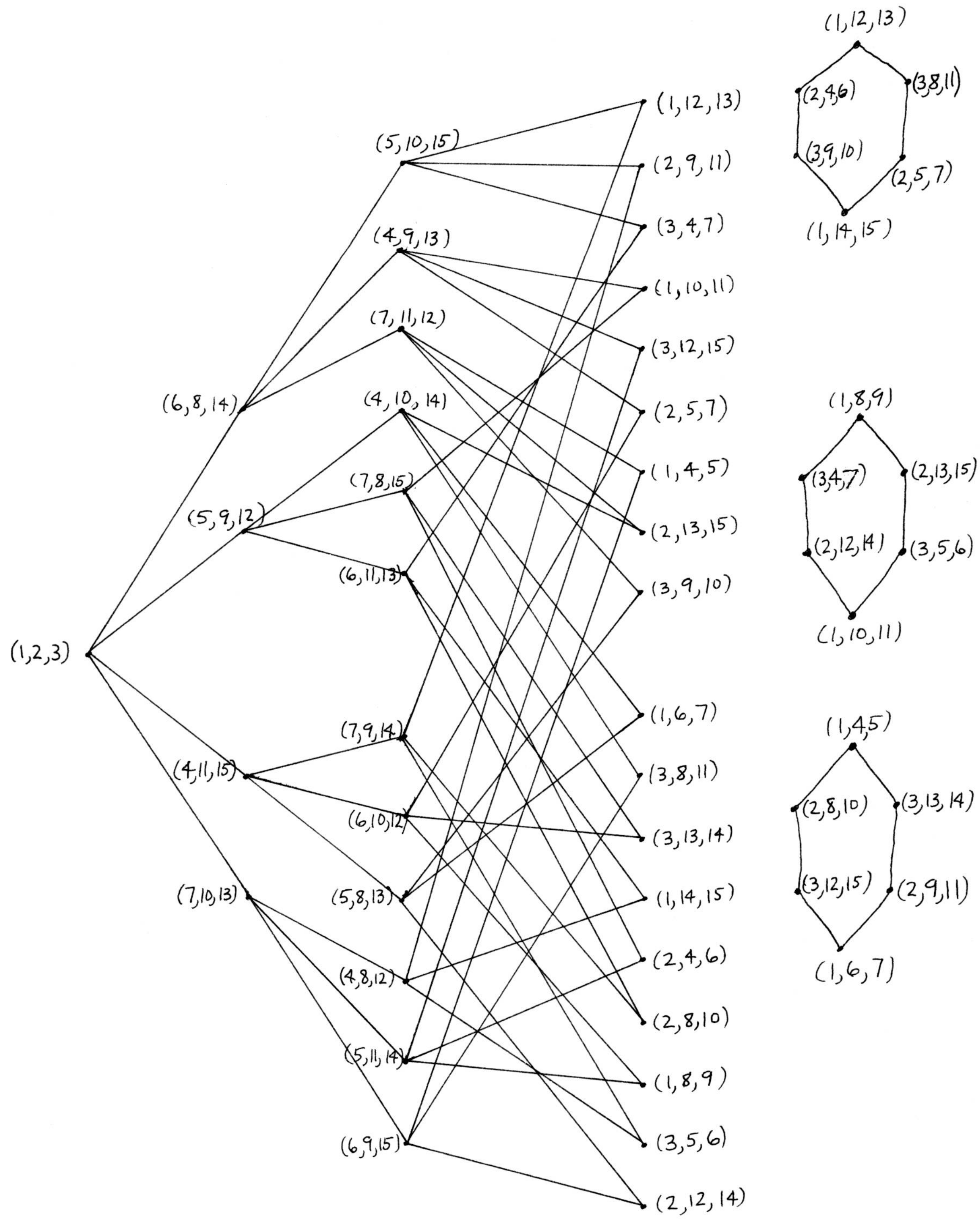

Figure 10. The Block Graph of the Hoffman-Singleton Graph.

code points and the sets $\{t\} \cup \{\Gamma_H(t)\}$ are called code spheres.

(6.2.4) Corollary. The graph $G_{\mathcal{B}}$ of (6.2.2) contains a perfect one-error correcting code.

Proof. The objects of the PG(3,2) design used in constructing $G_{\mathcal{B}}$ were written as $V = \{1,2, \ldots, 15\}$; we fix $i \in V$ and set $T_i = \{\beta \in \mathcal{B} : i \in \beta\}$. That the thirty-five blocks of PG(3,2) are partitioned in $G_{\mathcal{B}}$ by the code spheres of T_i follows from the fact that all β in T_i are mutually f_3-related. □

Note that if the unknown Moore graph $G = G(3250,57,0,1)$ were afforded by a design $D = D(400,2850,57,8,1)$ then there would be a perfect one-error correcting code by taking any fifty-seven blocks containing a common object. Indeed, such blocks would be at mutual distance three in $G_{\mathcal{B}}$ because of (6.2.1) and each of the fifty-seven code spheres would contain $1 + (57 - 8) = 50$ blocks, partitioning the 2850 blocks. The next theorem guarantees the existence of such a block graph, subject to design constructibility.

(6.2.5) Theorem. If $G = G(n,r,c,\lambda)$ is afforded by a quasi-symmetric design D whose intersection numbers are x and $y = \lambda$, such that blocks are adjacent only if they intersect in x objects, then the block configuration based on $\mathcal{O} = \{f_0 = I, f_1, f_2, f_3\}$, where $(\beta_i, \beta_j) \in f_1$ if and only if $|\beta_i \cap \beta_j| = x$, $\partial_G(\beta_i, \beta_j) = 1$, $(\beta_i, \beta_j) \in f_2$ if and only if $|\beta_i \cap \beta_j| = x$, $\partial_G(\beta_i, \beta_j) = 2$, and $(\beta_i, \beta_j) \in f_3$ if and only if $|\beta_i \cap \beta_j| = y$ is coherent, hence it is a distance regular graph of diameter three.

<u>Proof</u>. We know $\mathrm{diam}(G_{\mathcal{B}}) = 3$ because of (6.2.1). Certainly $\mathcal{O}$ partitions $G_{\mathcal{B}} \times G_{\mathcal{B}}$ and we have trivial pairing and homogeneity. For coherency it suffices by (2.1.5) to show that $Q\Gamma = \{\sum_{i=0}^{3} \mu_i \phi_{f_i} : \mu_i \in Q\}$ is a ring. If Y is the block intersection matrix of D we have

(6.2.6)(i) $$Y = kI + x(\phi_{f_1} + \phi_{f_2}) + y\phi_{f_3},$$

so $Y \in Q\Gamma$. We also have

(6.2.6)(ii) $$J = I + \phi_{f_1} + \phi_{f_2} + \phi_{f_3}.$$

Since $\phi_{f_1} = C$, the adjacency matrix of $G_{\mathcal{B}}$, and since $C^2 = (r - \lambda)I + (c - \lambda)C + \lambda J - Y$ (by (4.1.3)), we have $\phi_{f_1}^2 \in Q\Gamma$. We see by (6.2.6)(i), (ii) that

(6.2.7) $$\phi_{f_3} = \frac{1}{y-x}(Y - xJ + (x - k)I).$$

By $YJ = JY = rkJ$ and $Y^2 = k^2\lambda J + (r - \lambda)Y$ (see (2.4.7) and (2.4.8)) and (6.2.7) we have that $\phi_{f_3}^2 \in Q\Gamma$. Since

$$(6.2.8) \qquad \phi_{f_1} Y = Y\phi_{f_1} = (c - \lambda)Y + k\lambda J$$

(see (4.1.4)), (6.2.7) and (6.2.8) give us that $\phi_{f_3} \cdot \phi_{f_1} = \phi_{f_1} \cdot \phi_{f_3} \in Q\Gamma$. This suffices to establish that $Q\Gamma$ is a ring since all other products of the form $\phi_{f_i} \cdot \phi_{f_j}$ are expressible in terms of I, J, Y, ϕ_{f_1}. Since $\mathcal{O}$ gives a trivially paired, homogeneous, commutative, coherent configuration $G_{\mathcal{B}}$ is distance regular. □

More generally, we can prove

<u>(6.2.9) Theorem</u>. Let $G = G(n,r,c,\lambda)$ be afforded by D. The block graph $G_{\mathcal{B}}$ of G is distance regular of diameter three if and only if

(i) $C \circ Y = \alpha_1 C$ for some nonnegative integer $\alpha_1 < c$, and

(ii) for each block β we have $\Delta_{G_{\mathcal{B}}}(\beta) = S_\beta \dot{\cup} T_\beta$ (a disjoint union), where $\beta_i \in S_\beta$ if and only if $\partial_G(\beta_i,\beta) = 2$, $|\beta_i \cap \beta| = \alpha_2 < \lambda$, $\beta_i \in T_\beta$ if and only if $|\beta_i \cap \beta| = \lambda \neq \alpha_1$, (independent of β), and $|S_\beta|$, $|T_\beta|$ are constants independent of β, $|T_\beta| \neq 0$.

Proof. Assume $G_{\mathcal{B}}$ to be distance regular of diameter three, having adjacency relations $\mathcal{O} = \{I = f_0, f_1, f_2, f_3\}$ with $(\beta_i, \beta_j) \in f_\ell$ if and only if $\partial_{G_{\mathcal{B}}}(\beta_i, \beta_j) = \ell$. Note that $\phi_{f_1} = C$, the block adjacency matrix, so $\phi_{f_2} + \phi_{f_3} = J - C - I$. Now

(6.2.10) $$C(\phi_{f_2} + \phi_{f_3}) = \sum_{h \in \mathcal{O}} (a_{f_1 f_2 h} + a_{f_1 f_3 h}) \phi_h,$$

while

(6.2.11) $$C(\phi_{f_2} + \phi_{f_3}) = C(J - C - I) = (r - k)J - C^2 - C.$$

Combine $C^2 = (r - \lambda)I + (c - \lambda)C + \lambda J - Y$ with (6.2.11) and (6.2.10) to get

(6.2.12) $$\sum_{h \in \mathcal{O}} (a_{f_1 f_2 h} + a_{f_1 f_3 h}) \phi_h = (r - k - \lambda)J - (r - \lambda)I + (\lambda - c - 1)C + Y.$$

Take the Hadamard product of C with each side of (6.2.12) to get

(6.2.13) $$(a_{f_1 f_2 f_1} + a_{f_1 f_3 f_1})C = C(r - k - c - 1) + Y \circ C.$$

Hence

$$(6.2.14)\qquad Y \circ C = (a_{f_1f_2f_1} + a_{f_1f_3f_1} - r + k + c + 1)C$$

$$= \alpha_1 C,$$

establishing (i). Taking the Hadamard product of each side of (6.2.12) with ϕ_{f_i}, $i = 2, 3,$ gives

$$(6.2.15)\qquad \begin{cases} (a_{f_1f_2f_2} + a_{f_1f_3f_2})\phi_{f_2} = (r - k - \lambda)J + Y \circ \phi_{f_2}, \\ (a_{f_1f_2f_3} + a_{f_1f_3f_3})\phi_{f_3} = (r - k - \lambda)J + Y \circ \phi_{f_3}. \end{cases}$$

Hence, by (6.2.14) and (6.2.15) we have

$$(6.2.16)\qquad Y = kI + \alpha_1 C + \alpha_2\phi_{f_2} + \alpha_3\phi_{f_3},$$

where $\alpha_2 = a_{f_1f_2f_2} + a_{f_1f_3f_2} - r + k + \lambda$, $\alpha_3 = a_{f_1f_2f_3} + a_{f_1f_3f_3} - r + k + \lambda$. Since $\Delta_{G_{\mathcal{B}}}(\beta) = \{\beta_i \in \mathcal{B}: (\beta,\beta_i) \in f_2\} \dot{\cup} \{\beta_i \in \mathcal{B}: (\beta,\beta_i) \in f_3\}$ we need only establish that $\alpha_2 < \lambda = \alpha_3$.

If $(\beta_i,\beta_j) \in f_3$ then $\partial_G(\beta_i,\beta_j) = 2$ since $C \circ \phi_{f_3} = 0$. Hence there must be λ paths of length two in G between β_i and β_j. Yet in $G_{\mathcal{B}}$ there are 0 paths of length two between β_i and β_j, so $|\beta_i \cap \beta_j| = \lambda = \alpha_3$. Finally, for $(\beta_i,\beta_j) \in f_2$ we have $\partial_G(\beta_i,\beta_j) = 2$, hence $|\beta_i \cap \beta_j| = \alpha_2 \leq \lambda$. But equality would entail $(\beta_i,\beta_j) \in f_3$. Hence $\alpha_2 < \lambda$. This establishes (ii).

Conversely, suppose (i) and (ii) are in force. Define $\mathcal{O} = \{I = f_0, f_1, f_2, f_3\}$ on $G_{\mathcal{B}}$ by

$$(6.2.17)\quad \begin{cases} (\beta_i,\beta_j) \in f_1 & \text{if and only if } C_{ij} = 1, \\ (\beta_i,\beta_j) \in f_2 & \text{if and only if } \beta_i \in S_{\beta_j}\ (\beta_j \in S_{\beta_i}), \\ (\beta_i,\beta_j) \in f_3 & \text{if and only if } \beta_i \in T_{\beta_j}\ (\beta_j \in T_{\beta_i}). \end{cases}$$

Certainly $G_{\mathcal{B}}$ has diameter three since $|T_\beta| \neq 0$. Since $\mathcal{O}$ defines a homogeneous, trivially paired configuration partitioning $G_{\mathcal{B}} \times G_{\mathcal{B}}$, we need only show that $Q\Gamma = \{\sum_{i=0}^{3} \mu_i \phi_{f_i} : \mu_i \in Q\}$ is a ring.

As in the proof of (6.2.5) we have

$$(6.2.18)\quad \begin{cases} J - C - I = \phi_{f_2} + \phi_{f_3}, \\ Y = kI + \alpha_1 C + \alpha_2 \phi_{f_2} + \lambda \phi_{f_3}, \\ C^2 = (r - \lambda)I + (c - \lambda)C + \lambda J - Y, \\ CY = YC = (\lambda - c)Y + k\lambda J, \\ CJ = JC = (r - k)J, \\ Y^2 = k^2 \lambda J + (r - \lambda)Y, \\ YJ = JY = krJ. \end{cases}$$

Since J, $Y \in Q\Gamma$, and since (6.2.18) enables us to express ϕ_{f_2}, ϕ_{f_3} in terms of C, I, J, Y the coherency follows and the theorem is proved. □

6.3. Coherent Block Graphs

For non-multiple designs affording strongly regular graphs it is natural to expect the block graph to be "coherent" in the sense that the $G_{\mathcal{B}}$-distance between two blocks should depend only on their G-distance and intersection order. Thus we define the following notion:

(6.3.1) Definition. Let $D = D(v,b,r,k,\lambda)$ afford $G = G(n,r,c,\lambda)$ with D having an invariant intersection distribution and $m_k = 0$. Call (G,D) a coherently afforded pair if the configuration based on $\mathcal{O} = \{f_0 = I, f_i^j : i = 1,2,3; j = 1,2,3, \ldots, k-1\}$, where $(\beta_\ell, \beta_m) \in f_i^j$ if and only if $\partial_{G_{\mathcal{B}}}(\beta_\ell, \beta_m) = i$ and $|\beta_\ell \cap \beta_m| = j$, is coherent.

It can be verified that $G = G(56,10,0,2)$, $D = D(16,40,10,4,2)$ as discussed in (6.1.25) gives an example of a coherently afforded pair. Here we get $\mathcal{O} = \{I, f_1^0, f_2^0, f_2^1, f_3^2\}$, with subdegrees $n_I = 1$, $n_{f_1^0} = 6$, $n_{f_2^0} = 3$, $n_{f_2^1} = 24$, $n_{f_3^2} = 6$.

For future reference we state some pertinent counting formulas regarding coherently afforded pair.

<u>(6.3.2) Theorem.</u> If (G,D) is a coherently afforded pair then the subdegrees n_f satisfy

(i) $\sum_i i n_{f_1^i} = ck,$

(ii) $\sum_i n_{f_1^i} = r - k,$

(iii) $\sum_i n_{f_1^j} = m_j, \quad 1 \le j \le k - 1,$

(v) $\sum_i (\lambda - i) n_{f_2^i} = ck + (r - k)(r - k - c - 1).$

<u>Proof.</u> For (i), count the number of objects in $\Gamma_{G_{\mathcal{B}}}(\beta)$ which also lie in β. This number is the number of triangles on each of the object edges that emanate from β. For (ii), note that $\beta_\ell \sim \beta_m$ in $G_{\mathcal{B}}$ if and only if $(\beta_\ell, \beta_m) \in f_1^i$ for some i, hence (ii) follows. Part (iii) merely accounts for all blocks in $G_{\mathcal{B}}$ intersecting a given block in j objects. Finally, for (iv), $\lambda \sum_i n_{f_2^i}$ counts the number of G-paths of length two between a block β and all those blocks at distance two from β in $G_{\mathcal{B}}$.

But, counting another way, this value is the number of (block-object-block) paths plus the number of (block-block-block) paths, i.e., $\sum_i i n_{f_2^i} + \sum_i n_{f_1^i}(r - k - 1 - c + i)$. Equating these values and using (i), (ii) gives (iv). □

If (6.3.1) is not in force, for all choices of D, we would have doubts about the design constructibility of G, or at least we might conclude that block adjacencies would have to be rather bizarre. Thus (6.3.1) serves as a measure of the symmetry of G.

We conclude this chapter with a noncoherency result. The graph in question is unknown (Biggs, 1979, personal communication) and the coherency it lacks can be viewed as evidence against its possible existence.

(6.3.3) Theorem. No design D exists for which $G = G(99,14,1,2)$ and D form a coherently afforded pair.

Proof. By (3.2.3) a design D affording G would require parameters (22,77,14,4,2). Observe that D therefore has the invariant intersection distribution {30,40,6} by the Stanton-Sprott equations (4.22). Since each $\Gamma_{G_{\mathcal{B}}}(\beta)$ must appear as in Figure 11 and because of (6.3.2) we have $\mathcal{O} = \{f_0 = I, f_1^0, f_1^1, f_2^0, f_2^1, f_3^2\}$ with $n_I = 1$, $n_{f_1^0} = 6$, $n_{f_1^1} = 4$, $n_{f_2^0} = 24$, $n_{f_2^1} = 36$, and $n_{f_3^2} = 6$.

In attempting to fill out each $\hat{\phi}_f$ we make heavy use of the identity

$$(6.3.4) \qquad \Delta\hat{\phi}_g^T = \hat{\phi}_g\Delta, \qquad g \in \mathcal{O}$$

where $\qquad \Delta = \mathrm{diag}(1,6,4,24,36,6).$

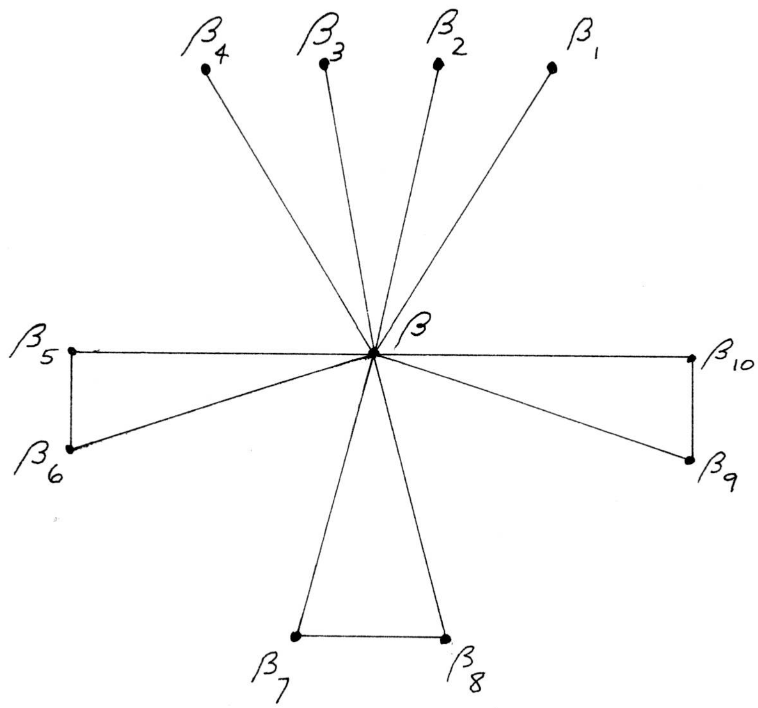

Figure 11. $\Gamma_{G_B}(\beta)$, $|\beta \cap \beta_i| = 1$, $1 \le i \le 4$, $|\beta \cap \beta_j| = 0$, $5 \le j \le 10$, $|\beta_i \cap \beta_j| = 0$, $5 \le i$, $j \le 10$.

Here we have a trivially paired, homogeneous, and hence commutativ, configuration. We will show that it cannot be coherent. Repeated application of (6.3.4), elementary counting arguments and the use of (2.1.4)(iii) as well as $a_{f_1^1 f_1^1 I} = 4$ gives

(6.3.5)
$$\hat{\phi}_{f_1^1} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 2 \\ 0 & 4 & 0 & 3-r & 4-q & 0 \\ 0 & 0 & 0 & r & q & 0 \\ 0 & 0 & 3 & 0 & 0 & 2 \end{pmatrix}$$

for r, q nonnegative integers such that $3(4-q) = 2r$. Since $(\hat{\phi}_{f_1^1})^2 = \sum_{h \in \mathcal{O}} a_{f_1^1 f_1^1 h} \hat{\phi}_h = 4I + 2\hat{\phi}_{f_3}$ we have

(6.3.6)
$$\hat{\phi}_{f_3^2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{3-r}{2} & 2-\frac{q}{2} & 0 \\ 0 & 0 & 3 & 0 & 0 & 2 \\ 0 & 6-2r & 0 & \frac{1}{2}((r-3)^2+r(4-q) & \frac{1}{2}(4-q)(3-r+q) & 0 \\ 0 & 2r & 0 & \frac{1}{2}(r(3-r+q)) & -2+\frac{1}{2}(r(4-q)+q^2) & 0 \\ 6 & 0 & 3 & 0 & 0 & 3 \end{pmatrix}.$$

Hence $q = 2$, $r = 3$ and

(6.3.7)
$$\hat{\phi}_{f_1^1} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 2 \\ 0 & 4 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 & 2 \end{pmatrix}.$$

Now, use (6.3.4) repeatedly, along with the fact that $\hat{\phi}_{f_1^0}\hat{\phi}_{f_1^1} = \hat{\phi}_{f_1^1}\hat{\phi}_{f_1^0}$, to obtain

$$(6.3.8) \qquad \hat{\phi}_{f_1^0} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 6 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 6 & 4-u & 5-t & 0 \\ 0 & 0 & 0 & u & t & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

where t, u are nonnegative integers for which $2u = 3(5 - t)$. Note that $a_{f_2^0 f_1^0 f_1^0} = 4$. We will show this to be impossible. Note that $a_{f_2^0 f_1^0 f_1^0}$ counts the number of blocks β_z such that $(\beta_z, \beta_x) \in f_2^0$ and $(\beta_z, \beta_y) \in f_1^0$, where $(\beta_x, \beta_y) \in f_1^0$ (see Figure 12). Since $c = 1$ there is a block β_μ such that (β_μ, β_x), $(\beta_\mu, \beta_y) \in f_1^0$. Without loss of generality, by appealing to Figure 11, we take $(\beta_{z_1}, \beta_{z_2}) \in f_1^0$ and $(\beta_{z_3}, \beta_{z_4}) \in f_1^0$. Hence we have Figure 13. Now $|\beta_x \cap \beta_j| = 0$ for $j = \mu, z_1, z_2, z_3, z_4$. Similarly, $|\beta_i \cap \beta_j| = 0$ for $i, j = x, \mu, z_1, z_2, z_3, z_4$, $i \neq j$. Thus β_μ, β_x, β_{z_1}, β_{z_2}, β_{z_3}, β_{z_4} account for $4 \cdot 6 = 24$ different objects, an impossibility since $v = 22$.

□

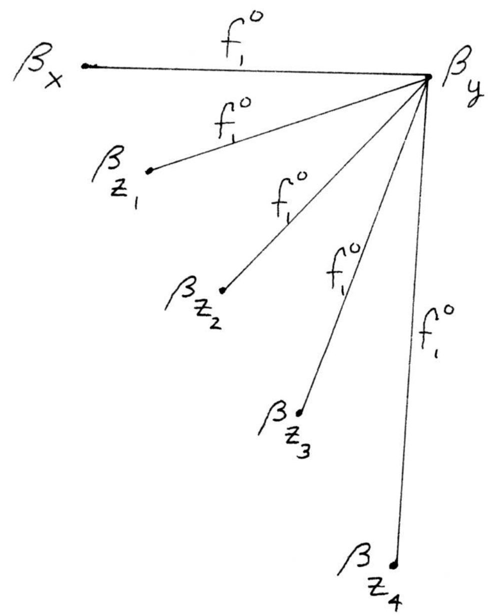

Figure 12. $|\beta_x \cap \beta_y| = 0$, $(\beta_x, \beta_{z_i}) \in f_2^0$, $1 \leq i \leq 4$.

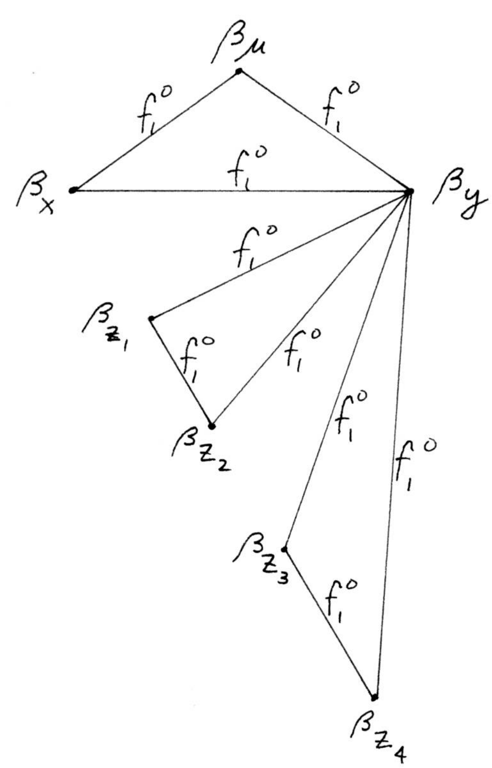

Figure 13. $a_{f_2^0 f_1^0 f_1^0}$.

CHAPTER 7

THE DODECAHEDRON AND FUSION MAPS

This chapter is devoted to the design construction of a strongly regular graph with parameters (26,10,3,4) as well as generalizations of the technique used to perform the construction. The original reason for examining this graph stems from a result of Biggs (1971) stating that there does not exist a distance transitive graph with those parameters. Thus we wish to determine the degree of symmetry of the graph if it is afforded by a design.

In 7.1 we give the design construction for such a graph and establish a close tie between the graph and the Petersen graph via a "fusion" of blocks. In 7.2 we are able to exploit the presence of the Petersen graph to determine the automorphism group of the graph and give it a pleasing geometric realization. Section 7.3 concludes the discussion by establishing the uniqueness of the graph in the sense of its "fusion" property.

7.1. The Design Construction

By (3.2.3) the design construction of a strongly regular graph $G = G(26,10,3,4)$ requires a design D with parameters (6,20,10,3,4). There are two natural choices for D, namely:

(7.1.1) (i) $D = K_3^6$,

(ii) D is obtained from two copies of the unique (see Hall, 1967) $\bar{D} = \bar{D}(6,10,5,3,2)$. Here we write $D = \bar{D}^2$.

Regarding the choice (7.1.1)(i) we have the next theorem.

<u>(7.1.2) Theorem.</u> K_3^6 does not coherently afford any $G = G(26,10,3,4)$.

<u>Proof.</u> Here we have, via (6.3.1), X consisting of twenty 3-sets from a 6-set and $\mathcal{O} = \{I, f_1^0, f_1^1, f_1^2, f_2^0, f_2^1, f_2^2\}$, with the possibility that some subdegrees may be 0. Note that K_3^6 has invariant block intersection distribution {1,9,9,0}, hence $\mathrm{diam}(G_{\mathcal{B}}) = 2$ because of (6.2.1). From (6.3.2) we get

$$\sum_i i n_{f_1^i} = 9,$$

$$\sum_i n_{f_1^i} = 7,$$

$$\sum_i n_{f_i^0} = 1, \qquad \sum_i n_{f_i^1} = 9 = \sum_i n_{f_i^2},$$

and

$$\sum_i (4 - i) n_{f_2^i} = 30.$$

Thus we must take

(7.1.3) $$n_I = 1, \quad n_{f_1^0} = 1, \quad n_{f_1^1} = 3,$$

$$n_{f_1^2} = 3, \quad n_{f_2^1} = 6, \quad n_{f_2^2} = 6$$

giving us (as in 2.1)

(7.1.4) $$\Delta = \mathrm{diag}(1,1,3,3,6,6).$$

Because of trivial pairing on $\mathcal{O}$ we will be exploiting (2.1.7), now written as

(7.1.5) $$\Delta(\hat{\phi}_g)^T = \hat{\phi}_g \Delta, \qquad g \in \mathcal{O}.$$

Using the requirement of (2.1.4)(iii), the definition $n_f = a_{ffI}$ of subdegree, and elementary counting arguments we initially obtain

$$(7.1.6)\qquad \hat{\phi}_{f_1^0} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y & 0 & w \\ 0 & 0 & x & 0 & z & 0 \\ 0 & 0 & 0 & 1-y & 0 & 1-w \\ 0 & 0 & 1-x & 0 & 1-z & 0 \end{pmatrix}$$

for $x, y, z, w \in \{0,1\}$. Applying (7.1.5) to (7.1.6) yields

$$(7.1.7)\qquad \hat{\phi}_{f_1^0} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

For $\hat{\phi}_{f_1^1}$ we use (2.1.6), (2.1.2)(ii), and (7.1.7) to argue that

$$(7.1.8)\qquad \hat{\phi}_{f_1^0} \cdot \hat{\phi}_{f_1^1} = \sum_{g\in \mathcal{O}} a_{f_1^0 f_1^1 g}\hat{\phi}_g = \sum_{g\in \mathcal{O}} a_{f_1^1 f_1^0 g}\hat{\phi}_g$$

$$= \hat{\phi}_{f_1^1} \hat{\phi}_{f_1^0}.$$

By the same techniques used to obtain $\hat{\phi}_{f_1^0}$ we have initially

$$(7.1.9) \qquad \hat{\phi}_{f_1^1} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & a & b & c & d & e \\ 0 & f & g & h & i & j \\ 0 & k & \ell & m & n & p \\ 0 & q & r & s & t & u \end{pmatrix},$$

where all entries are nonnegative integers and all column sums are 3. Apply (7.1.8) to obtain the conditions: $b = h$, $n = u$, $c = g$, $p = t$, $a = 0$, $f = 3$, $i = e$, $d = j$, $q = k = 0$, $r = m$, $\ell = s$.

The block adjacency matrix C is in Γ the adjacency ring of $(X,\mathcal{O})$ since $C = \sum_{i=0}^{2} \phi_{f_1^i}$. Hence $\hat{C} \in \hat{\Gamma}$, with $\hat{C} = \sum_{i=0}^{2} \hat{\phi}_{f_1^i}$. Hence $tr(\hat{C})$ is the sum of the distinct eigenvalues of C. By (4.1.2) and (2.3.8) we have

$$\text{Spec}(C) = \begin{pmatrix} 7 & -3 & 2 & -1 \\ \\ 1 & 6 & 8 & 5 \end{pmatrix}.$$

Thus $tr(\hat{C}) = 5$. From (7.1.8) and (7.1.7) we have

$\hat{\phi}_{f_1^1} \cdot \hat{\phi}_{f_1^0} = \hat{\phi}_{f_1^2}$ and so, by (7.1.7), (7.1.9) and the relations on the entries of $\hat{\phi}_{f_1^1}$ we must have

$$5 = tr(\hat{\phi}_{f_1^0} + \hat{\phi}_{f_1^1} + \hat{\phi}_{f_1^0} \cdot \hat{\phi}_{f_1^1}) = 0 + 2h + 2n + 2c + 2p,$$

an impossibility. □

With the negative information provided by the previous theorem, we choose $\bar{D}^2$ for design construction of $G = G(26,10,3,4)$.

<u>(7.1.10) Theorem</u>. The design $\bar{D}^2$ affords a strongly regular graph $G_{\bar{D}^2}(26,10,3,4)$.

<u>Proof</u>. Represent the blocks of $\bar{D}^2$ as

$$\{(1,2,3)^0,(1,2,5)^0,(2,3,6)^0,(2,4,6)^0,(1,3,4)^0,$$
$$(3,4,5)^0,(2,4,5)^0,(3,5,6)^0,(1,4,6)^0,(1,5,6)^0,$$
$$(1,2,3)^1,(1,2,5)^1,(2,3,6)^1,(2,4,6)^1,(1,3,4)^1,$$
$$(3,4,5)^1,(2,4,5)^1,(3,5,6)^1,(1,4,6)^1,(1,5,6)^1\}.$$

The superscripts, indicating the respective copies, are taken modulo 2.

Since we will be using two identical copies of $\bar{D}$, we note that there is no contradiction in agreeing that each block is adjacent to the block identical to it. This amounts to putting three triangles (from objects) on each edge $\langle \beta^0, \beta^1 \rangle$. Having agreed to this convention, we cannot have a block β^σ ($\sigma = 0$ or 1) adjacent to two identical blocks (since otherwise there would be four triangles on the resulting edge). Hence $\Gamma_{G_{\mathcal{B}}}(\beta^0) \cap \Gamma_{G_{\mathcal{B}}}(\beta^1) = \emptyset$ for every $\beta^0 \in \bar{D}$. The rule that $\beta^0 \sim \beta^1$ for every $\beta^0 \in \bar{D}$ is almost sufficient to determine the block adjacency rule. Since $\Gamma_{G_{\mathcal{B}}}(\beta^\sigma)$ must contain four repetitions of each of the objects not in β^σ and three repetitions of each of the objects in β^σ, (see (4.1.13)(ii)), the following rule determines the makeup of the blocks of $\Gamma_{G_{\mathcal{B}}}(\beta^\sigma)$:

(7.1.11) For $\beta^\sigma \in \bar{D}^2$, where $\beta^\sigma = (i,j,k)^\sigma$, we take $\Gamma_{G_{\mathcal{B}}}(\beta^\sigma)$ to consist of $\beta^{\sigma+1}$ plus the six uniquely determined (up to copy) blocks $(\ell,m,i)^{\sigma_1}$, $(\ell,m,j)^{\sigma_2}$, $(\ell,n,i)^{\sigma_3}$, $(\ell,n,k)^{\sigma_4}$, $(m,n,j)^{\sigma_5}$, $(m,n,k)^{\sigma_6}$, where ℓ, m, n are the three objects not in β^σ.

Here, at least, is a design for which the block adjacency rule can (up to copy) be uniquely prescribed, independent of the block. Thus, in a sense, the resulting block graph is coherent. For example, with $\beta^0 = (1,2,3)^0$ we get

$\Gamma_{G_{\mathcal{B}}}(\beta^0) = \{\beta^1, (3,4,5)^{\sigma_1}, (2,4,5)^{\sigma_2}, (1,4,6)^{\sigma_3}, (2,4,6)^{\sigma_4},$ $(3,5,6)^{\sigma_5}, (1,5,6)^{\sigma_6}\}$. The only undecided part that must be settled is the selection of $\sigma_1, \sigma_2, \ldots, \sigma_6$. This is where (4.1.13)(iii) is used. Note that (7.1.11) requires

$$\text{(7.1.12)} \qquad \beta^\sigma \sim \beta_i^\tau \text{ if and only if } \beta^{\sigma+1} \sim \beta_i^{\tau+1}$$

which cuts the work in half. Also, (7.1.12) is equivalent to saying that the map (involution) $\theta : \beta^\sigma \to \beta^{\sigma+1}$ defined on $\bar{D}^2$ is an automorphism of G. We may write the block intersection matrix Y of $\bar{D}^2$ as

$$\text{(7.1.13)} \qquad Y = \begin{bmatrix} \bar{Y} & \bar{Y} \\ \bar{Y} & \bar{Y} \end{bmatrix},$$

where $\bar{Y}$ is the block intersection matrix of $\bar{D}$, and C may therefore be written as

$$\text{(7.1.14)} \qquad C = \begin{bmatrix} C_1 & C_2 \\ C_2 & C_1 \end{bmatrix},$$

since $\Gamma_{G_{\mathcal{B}}}(\beta^{\sigma+1})$ is determined by $\Gamma_{G_{\mathcal{B}}}(\beta^\sigma)$. We also have that $C_1 \circ C_2 = 0$. There is enough information now to limit significantly the choices in the makeup of $\Gamma_{G_{\mathcal{B}}}(\beta^\sigma)$.

To begin the construction we take $\Gamma_{G_{\mathcal{B}}}((1,2,3)^0)$ $= \{(1,2,3)^1, (1,4,6)^1, (1,5,6)^1, (2,4,5)^1, (2,4,6)^0, (3,5,6)^0, (3,4,5)^1\}$ without loss of generality. This also determines $\Gamma_{G_{\mathcal{B}}}((1,2,3)^1)$ because of (7.1.12). This choice yields Figure 14.

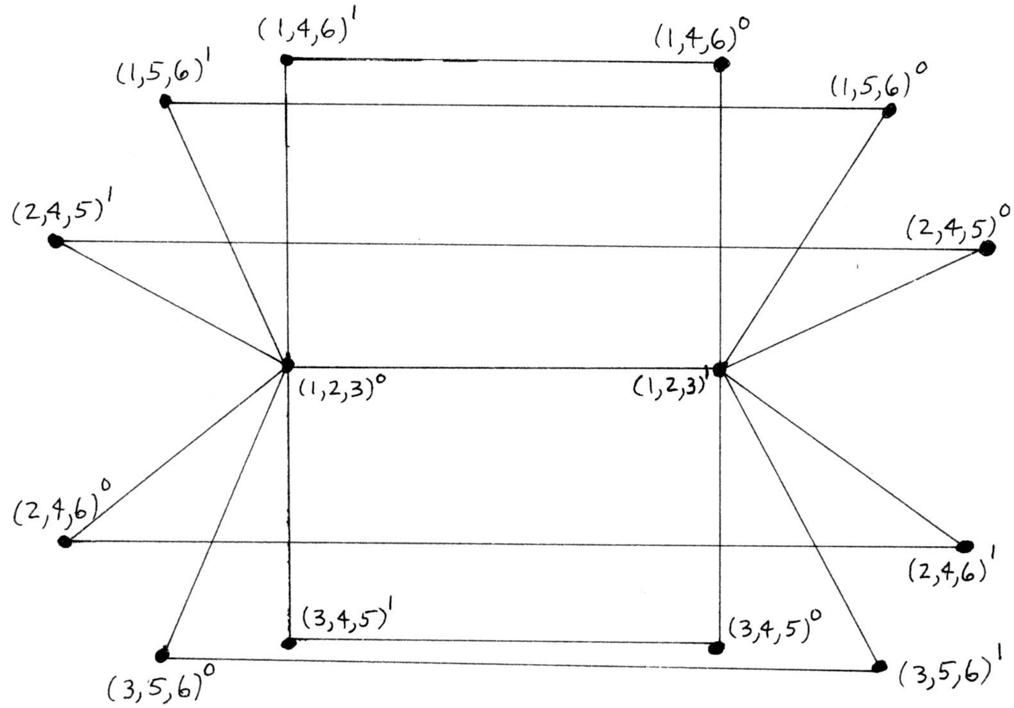

Figure 14. $\Gamma_{G_{\mathcal{B}}}((1,2,3)^{\sigma})$, $\sigma = 0, 1$.

Consider $\Gamma_{G_{\mathcal{B}}}((1,4,6)^1)$. It must consist of $(1,4,6)^0$ and $(1,2,3)^0$ and, up to copy choice, $(1,2,5)^{\sigma_1}$, $(4,3,5)^{\sigma_2}$, $(4,2,5)^{\sigma_2}$, $(6,3,5)^{\sigma_4}$, $(6,2,3)^{\sigma_5}$. From (4.1.13)(iii) applied to the edge $\langle(1,2,3)^0,(1,4,6)^1\rangle$ we see that $(1,4,6)^1$ is adjacent to two of the blocks of $\Gamma_{G_{\mathcal{B}}}((1,2,3)^0) \setminus (1,2,3)^1$ i.e., two of $(2,4,5)^1$, $(3,5,6)^0$, $(3,4,5)^1$. Applying (4.1.13)(iii) to $(1,2,3)^0$ and to some block of $\Gamma_{G_{\mathcal{B}}}((1,2,3)^1)$ we see that an additional path of length two is needed between $(1,2,3)^0$ and each of the blocks of $\Gamma_{G_{\mathcal{B}}}((1,2,3)^1)$. Hence $(1,4,6)^1$, for example, is adjacent to one of $(2,4,5)^0$, $(3,5,6)^1$, or $(3,4,5)^0$. There are analogous requirements for $\Gamma_{G_{\mathcal{B}}}(\beta)$ for each $\beta^\tau \in \Gamma_{G_{\mathcal{B}}}((1,2,3)^0)\setminus(1,2,3)^1$. In the interest of symmetry we choose the two blocks of $\Gamma_{G_{\mathcal{B}}}((1,2,3)^0)\setminus(1,2,3)^1$ to be connected with $(1,4,6)^1$ as that pair having but one common object. Proceeding in this manner we obtain:

$$(7.1.15)\quad \begin{cases} \Gamma_{G_{\mathcal{B}}}((1,4,6)^1) \supseteq \{(2,4,5)^1,(3,5,6)^0,(3,4,5)^0\}, \\ \Gamma_{G_{\mathcal{B}}}((1,5,6)^1) \supseteq \{(2,4,6)^0,(3,4,5)^1,(2,4,5)^0\}, \\ \Gamma_{G_{\mathcal{B}}}((2,4,5)^1) \supseteq \{(1,4,6)^1,(3,5,6)^0,(1,5,6)^0\}, \\ \Gamma_{G_{\mathcal{B}}}((2,4,6)^0) \supseteq \{(1,5,6)^1,(3,4,5)^1,(3,5,6)^1\}, \\ \Gamma_{G_{\mathcal{B}}}((3,5,6)^0) \supseteq \{(1,4,6)^1,(2,4,5)^1,(2,4,6)^1\}, \\ \Gamma_{G_{\mathcal{B}}}((3,4,5)^1) \supseteq \{(1,5,6)^1,(2,4,6)^0,(1,4,6)^0\}. \end{cases}$$

We have now nearly completed fourteen rows of C. Applying (4.1.3)(i) repeatedly to these rows and the remaining rows of C, and keeping in mind the rules (7.1.11) and (7.1.12) we eventually obtain the block adjacency matrix listed in (7.1.16) which appears on page 111. The graph obtained is unique up to the switching of labels on any pair of identical blocks.

Taking the 6×10 object-block incidence matrix $\bar{B}$ of $\bar{D}$ to get $B = [\bar{B}\ \bar{B}]$ for the object block incidence matrix of $\bar{D}^2$ we observe that

$$A = \begin{pmatrix} 0 & B \\ & \\ B^T & C \end{pmatrix}$$

satisfies (2.3.4) and (2.3.5). Hence we have a design construction for a strongly regular graph, which we will denote by $G_{\bar{D}^2} = G_{\bar{D}^2}(26,10,3,4)$. □

It would be gratifying to have a more pleasing picture of $G_{\bar{D}^2}$ than its adjacency matrix. To that end we introduce the next definition.

<u>(7.1.17) Definition</u>. Let $D = D(v,2b,2r,k,2\lambda)$ be a design of multiplicity two obtained from two copies of $D' = D'(v,b,r,k,\lambda)$. Label the blocks of D as β^{σ} with

(7.1.16)

	$(1,2,3)^0$	$(1,2,5)^0$	$(2,3,6)^0$	$(2,4,6)^0$	$(1,3,4)^0$	$(3,4,5)^0$	$(2,4,5)^0$	$(3,5,6)^0$	$(1,4,6)^0$	$(1,5,6)^0$	$(1,2,3)^1$	$(1,2,5)^1$	$(2,3,6)^1$	$(2,4,6)^1$	$(1,3,4)^1$	$(3,4,5)^1$	$(2,4,5)^1$	$(3,5,6)^1$	$(1,4,6)^1$	$(1,5,6)^1$
$(1,2,3)^0$	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0	1	1	0	1	1
$(1,2,5)^0$	0	0	0	1	0	1	0	0	0	0	0	1	1	0	1	0	0	1	1	0
$(2,3,6)^0$	0	0	0	0	0	0	0	0	1	1	0	1	1	0	1	1	1	0	0	0
$(2,4,6)^0$	1	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	1	0	1
$(1,3,4)^0$	0	0	0	0	0	0	1	1	0	0	0	1	1	1	1	0	0	0	0	1
$(3,4,5)^0$	0	1	0	0	0	0	0	0	0	1	1	0	1	1	0	1	0	0	1	0
$(2,4,5)^0$	0	0	0	0	1	0	0	0	1	0	1	0	1	0	0	0	1	1	0	1
$(3,5,6)^0$	1	0	0	0	1	0	0	0	0	0	0	1	0	1	0	0	1	1	1	0
$(1,4,6)^0$	0	0	1	0	0	0	1	0	0	0	1	1	0	0	0	1	0	1	1	0
$(1,5,6)^0$	0	0	1	0	0	1	0	0	0	0	1	0	0	1	1	0	1	0	0	1
$C = (1,2,3)^1$	1	0	0	0	0	1	1	0	1	1	0	0	0	1	0	0	0	1	0	0
$(1,2,5)^1$	0	1	1	0	1	0	0	1	1	0	0	0	0	1	0	1	0	0	0	0
$(2,3,6)^1$	0	1	1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	1	1
$(2,4,6)^1$	0	0	0	1	1	1	0	1	0	1	1	1	0	0	0	0	0	0	0	0
$(1,3,4)^1$	0	1	1	1	1	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0
$(3,4,5)^1$	1	0	1	1	0	1	0	0	1	0	0	1	0	0	0	0	0	0	0	1
$(2,4,5)^1$	1	0	1	0	0	0	1	1	0	1	0	0	0	0	1	0	0	0	1	0
$(3,5,6)^1$	0	1	0	1	0	0	1	1	1	0	1	0	0	0	1	0	0	0	0	0
$(1,4,6)^1$	1	1	0	0	0	1	0	1	1	0	0	0	1	0	0	0	1	0	0	0
$(1,5,6)^1$	1	0	0	1	1	0	1	0	0	1	0	0	1	0	0	1	0	0	0	0

$\sigma = 0, 1$ denoting respective copy. Let D afford $G = G(n,2r,c,2\lambda)$. Let $\theta:\beta^{\sigma} \to \beta^{\sigma+1}$ be a block graph automorphism and let $\beta_i^0 \sim \beta_i^1$ for all i. Define the <u>fusion subgraph</u> G_f of $G_{\mathcal{B}}$ as that graph whose vertices β are identified as block pairs $\beta = (\beta^0,\beta^1)$; β^0, β^1 in $G_{\mathcal{B}}$. The rule of adjacency in G_f is

$$\Gamma_{G_f}(\beta)$$

$$= \{\bar{\beta} = (\bar{\beta}^0,\bar{\beta}^1) \in G_f:\beta^{\sigma} \sim \bar{\beta}^{\tau}, \sigma = 0 \text{ or } 1, \tau = 0 \text{ or } 1\}.$$

<u>(7.1.18) Theorem</u>. Let (G,D) be as described in (7.1.17). The block adjacency matrix C of $G_{\mathcal{B}}$ may be written

$C = \begin{pmatrix} C_1 & C_2 \\ C_2 & C_1 \end{pmatrix}$ and the adjacency matrix of G_f is $C_1 + C_2 - I - C_1 \circ C_2$. In particular, for $c = k$ the adjacency matrix is $C_1 + C_2 - I$ and G_f has valence $2r - k - 1$.

<u>Proof</u>. Requiring that $\theta:\beta^{\sigma} \to \beta^{\sigma+1}$ be a $G_{\mathcal{B}}$ automorphism is equivalent to imposing the rule $\beta^{\sigma} \sim \beta_i^{\tau}$ if and only if $\beta^{\sigma+1} \sim \beta_i^{\tau+1}$, so clearly C has the form indicated. From the presence of θ we note that β^0, $\beta^1 \sim \bar{\beta}^{\sigma}$ entails β^0, $\beta^1 \sim \bar{\beta}^{\sigma+1}$. Thus, if $\Gamma_{G_{\mathcal{B}}}(\beta_i^0) \cap \Gamma_{G_{\mathcal{B}}}(\beta_i^1) = \{\beta_{i_1}^0, \beta_{i_1}^1, \ldots, \beta_{i_p}^0, \beta_{i_p}^1\}$, say, then $(C_1 + C_2)_{ii_m} = 2$, $1 \le m \le p$, and so $(C_1 \circ C_2)_{ii_m} = 1$, $1 \le m \le p$. Note also that C_2 has

$(C_2)_{ii} = 1$, $1 \leq i \leq b$. Thus G_f has the indicated adjacency matrix.

In case $c = k$ then every edge $\langle \beta^0, \beta^1 \rangle$ of G_B has no triangles, whence $C_1 \circ C_2 = 0$. Thus $C_1 + C_2 - I$ is the correct G_f adjacency matrix in this case. Clearly, the valence in G_f is $2r - k - 1$ in case $c = k$. □

We now establish the relationship between $G_{\bar{D}^2}$ and the Petersen graph.

(7.1.19) Theorem. For $G_{\bar{D}^2}$, the fusion subgraph G_f is the complement of the Petersen graph having parameters (10,6,3,4).

Proof. We have (7.1.17) in force and $c = k$. From (7.1.16) we verify that $C_1 + C_2 - I = C_f$, the adjacency matrix of G_f, satisfies $C_f J = 6J$ and $C_f^2 = 2I - C_f + 4J$. Hence, by (2.3.4) and (2.3.5), G_f is strongly regular with parameters (10,6,3,4) and hence is the complement of the Petersen graph. □

7.2. The Dodecahedron

From the fusion subgraph G_f obtained above we can realize $G_{\bar{D}^2}$ geometrically. To motivate the next theorem we make the following observation:

<u>(7.2.1) Proposition</u>. A Petersen graph can be obtained from the ten antipodal pairs of vertices of a dodecahedron.

<u>Proof</u>. Recall that a dodecahedron has twenty pentagonal faces, twenty vertices, twenty edges. Identify an antipodal pair of vertices as one fused vertex and construct a graph on the ten fused vertices by adjoining two fused vertices if and only if the corresponding original four dodecahedral vertices are the vertices on two dodecahedral edges. Clearly the result is a Petersen graph. □

<u>(7.2.2) Theorem</u>. $G_{\bar{D}^2}$ may be constructed from the faces and vertices of a dodecahedron.

<u>Proof</u>. The six objects of the design $\bar{D}$ are the six antipodal pairs of pentagons on the dodecahedron. Label the (fused) faces $\{1,2,3,4,5,6\}$. The twenty blocks of the multiple design are the twenty vertices of the dodecahedron where each block contains those objects which, as pentagons, meet at the given block vertex. See Figure 15.

We see that a copy of the ten blocks of $\bar{D}$ is obtained by taking the five blocks forming any pentagon and the five further blocks that are dodecahedrally adjacent to these (much like a "top half" of the dodecahedron). The other copy of $\bar{D}$ is made up of the remaining ten corners (antipodal to the first ten). Note that antipodal corners

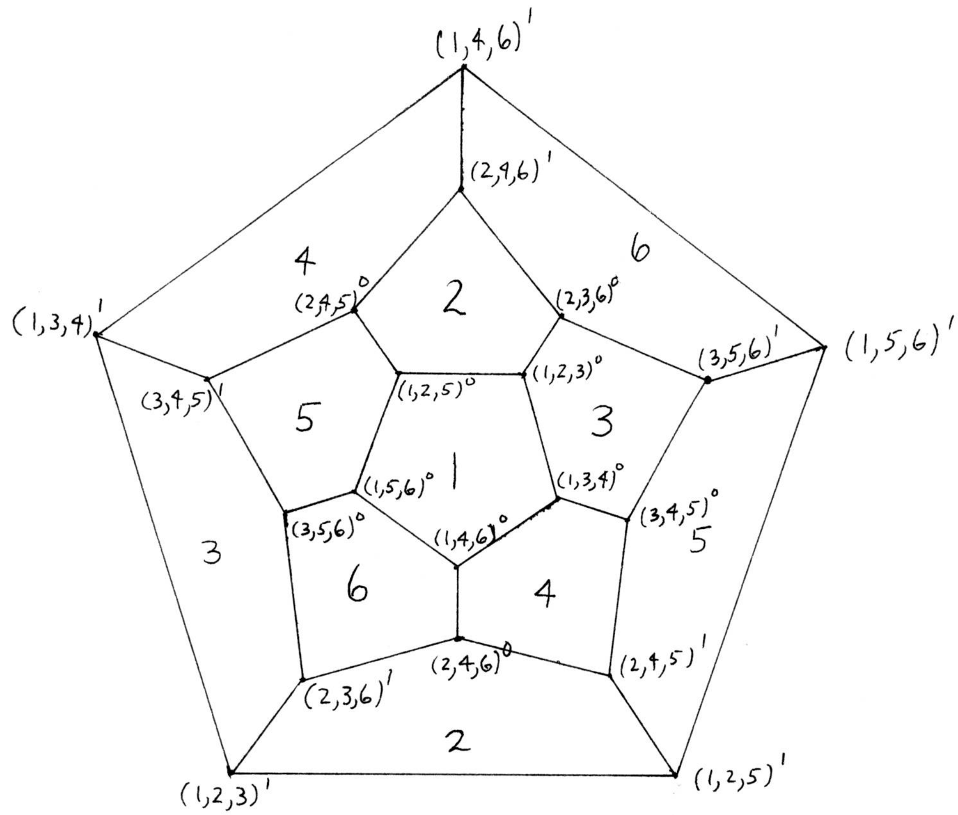

Figure 15. The Dodecahedron, Embedded in the Plane. (Object labeling done so as to be consistent with $\bar{D}$.)

correspond to identical blocks. Our design construction of $G_{\bar{D}^2}$ is then obtained from the dodecahedron as follows:

> Taking the six objects (antipodal pentagon pairs) and twenty blocks for the twenty-six vertices of $G_{\bar{D}^2}$, we connect an object vertex to the ten blocks (dodecahedral corners) bordering the corresponding pentagonal pair. We connect a block β^σ to the three objects that meet it on the dodecahedron, to the block

$\beta^{\sigma+1}$ antipodal to β^{σ}, and to those six blocks which, as dodecahedral corners, are at distance three from β^{σ} on the dodecahedron.

This rule, in fact, gives the adjacency matrix A as described in Section 7.1. □

By means of the dodecahedral realization we have the next theorem.

<u>(7.2.3) Theorem.</u>

(i) $\mathrm{Aut}(G_{\bar{D}^2}) \cong \mathcal{I}$, the Icosahedral group,

(ii) $\mathrm{Aut}(\bar{D}) \cong A_5$,

(iii) $\mathrm{Aut}(\bar{D}^2) \cong A_5 \times (Z_2)^{10}$,

(iv) The map that fuses $(G_{\bar{D}^2})_{\mathcal{B}}$ into G_f induces a homomorphism $\phi: \mathcal{I} \to \mathrm{Aut}(P)$ where $\ker(\phi) \cong Z_2$ and $\mathrm{Aut}(P)$ is the automorphism group of the Petersen graph; hence $A_5 < \mathrm{Aut}(P)$.

<u>Proof.</u> It is well known (see Benson and Grove, 1971) that the group of symmetries of a regular dodecahedron is $\mathcal{I} \cong A_5 \times Z_2$, A_5 being the rotation subgroup and Z_2 the subgroup that interchanges antipodal dodecahedral vertices. Thus we have (i).

For (ii) we may identify antipodal dodecahedral vertices as the <u>same</u> block. From our embedding of $\bar{D}$ in a dodecahedron we have that $Aut(\bar{D})$ consists of precisely those object permutations (pentagon permutations) that preserve the dodecahedron. Hence $Aut(\bar{D})$ is isomorphic to the rotation subgroup of I, i.e., $Aut(\bar{D}) \cong A_5$.

For (iii) observe that the switching of the labels on any pair (β^0, β^1) of blocks in $\bar{D}^2$ preserves $\bar{D}^2$ so (iii) follows.

For (iv) we note that for a permutation of the vertices (blocks) resulting from a rotation $\psi \in A_5$ of the dodecahedron we have $\psi(\beta_i^\sigma) = \beta_j^\tau$ if and only if $\psi(\beta_i^{\sigma+1}) = \beta_j^{\tau+1}$. Thus any rotation may be represented by a permutation matrix of the form $\begin{pmatrix} R_1 & R_2 \\ R_2 & R_1 \end{pmatrix}$ and hence any matrix Q representing an element of I is of the form

$$Q = \begin{pmatrix} R_1 & R_2 \\ R_2 & R_1 \end{pmatrix} \quad \text{or} \quad Q = \begin{pmatrix} R_1 & R_2 \\ R_2 & R_1 \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} R_2 & R_1 \\ R_1 & R_2 \end{pmatrix}.$$

Define $\phi: I \to Aut(P)$ by setting $\phi(Q) = (R_1 + R_2)$ clearly ϕ is a homomorphism for which

$$\phi(Q)(C_1 + C_2 - I) = (C_1 + C_2 - I)\phi(Q)$$

if and only if $QC = CQ$. Also, $\ker(\phi) = \left\langle \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, (I) \right\rangle$.

Therefore $I/\ker(\phi) \cong A_5 < \mathrm{Aut}(P)$. □

All strongly regular graphs having parameters (26,10,3,4) were classified by Weisfeiler (1976) by means of a computer search.

Among the strongly regular graphs with parameters (26,10,3,4), another graph that was constructed by Biggs (1974) deserves mention because it exhibits the phenomenon of (3.3.8)(iv). If a strongly regular $G = G(26,10,3,4)$ contains a maximal strongly regular type $(s - k)$ eigengraph Ω, one may verify that $|\Omega| = 13$ and that the valence of Ω must be six. The only possibility is $\Omega = \Omega(13,6,2,3)$. There is such a graph Ω, called the Paley graph $P(13)$ (see Cameron and Van Lint, 1975) which is defined as follows:

> The thirteen vertices are the thirteen residues mod 13. Two residues are adjacent if and only if their difference is a nonzero square.

It is easily checked that the graph constructed by Biggs can be partitioned into two $P(13)$ eigengraphs. It is also a routine matter to verify that this graph is afforded by a twisted design formed from one copy of $\bar{D}$ together with ten blocks that are obtained by switching two object labels in $\bar{D}$, and three objects come from each P(13) eigengraph.

7.3. Fusion Strongly Regular Graphs

We ask next whether there exist any other design constructible strongly regular graphs whose block adjacencies are similar to those of the dodecahedral graph $G_{\bar{D}^2}$, or whether there are any strongly regular d.c. graphs that have strongly regular fusion subgraphs.

(7.3.1) Definition. Call $G = G(n,2r,c,2\lambda)$ fusion strongly regular if

(i) G is as in (7.1.17), and

(ii) the fusion of identical blocks induces a strongly regular fusion subgraph.

Note that the dodecahedral graph is fusion strongly regular. Are there other examples of fusion strongly regular graphs? For simplicity we consider the case when $c = k$.

(7.3.2) Theorem. In order that $G = G(n,2r,c,2\lambda)$, with $c = k$, be fusion strongly regular via two copies of $\bar{D} = \bar{D}(v,b,r,k,\lambda)$ it is necessary that $\bar{D}$ be quasi-symmetric with intersection orders $x = k/(2k-2\lambda+1)$, $y = \lambda/(k-\lambda)$, and have $v = (k^2(k-1) + \lambda(2k-k^2))/\lambda$, $b = (k^2 - k\lambda + \lambda)(k^2 - k\lambda - k + 2\lambda)/\lambda$ and $r = k^2 - k\lambda + \lambda$.

Proof. We first derive the parameters $(n_f, r_f, c_f, \lambda_f)$ of G_f, the fusion subgraph of G_B. We have already seen that $n_f = b$, $r_f = 2r - k - 1$. Label the blocks of two copies of $\bar{D}$ as $\{\beta_1^0, \beta_2^0, \ldots, \beta_b^0; \beta_1^1, \beta_2^1, \ldots, \beta_b^1\}$, superscripts being taken modulo 2 to distinguish the respective copies. Figure 16 represents a typical situation in G_B.

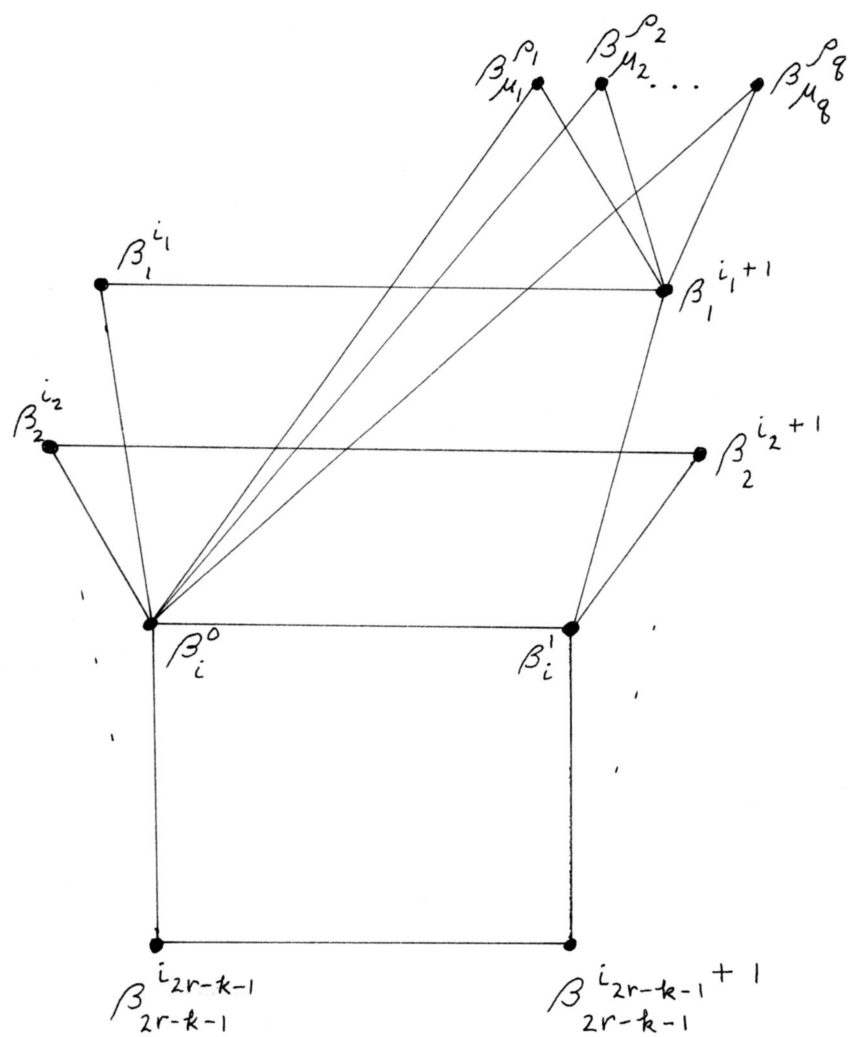

Figure 16. $\Gamma_{G_B}(\beta_i^\sigma)$ $\sigma = 0, 1$.

When identical blocks are fused to form G_f, the edge $\langle \beta_i^0, \beta_1^{i_1} \rangle$ obtains $k - |\beta_i^0 \cap \beta_1^{i_1}|$ triangles from the triangles in $G_{\mathcal{B}}$ on edge $\langle \beta_i^0, \beta_i^{i_1} \rangle$ together with q triangles from the vertices $\beta_{\mu_1}^{\rho_1}, \ldots, \beta_{\mu_q}^{\rho_q}$ in $\Gamma_{G_{\mathcal{B}}}(\beta_i^0) \cap \Gamma_{G_{\mathcal{B}}}(\beta_1^{i_1+1})$. We see that $q = 2\lambda - 2 - |\beta_i^0 \cap \beta_1^{i_1}|$. Thus $c_f = k - 2|\beta_i^0 \cap \beta_1^{i}| + 2\lambda - 2$, and we see that $|\beta_i^0 \cap \beta_1^{i_j}| = |\beta_i^0 \cap \beta_1^{i_1}|$, $2 \leq j \leq 2r - k - 1$ and this common number is independent of i. Calling the common intersection value x we have

(7.3.3) $$c_f = k + 2\lambda - 2x - 2.$$

From $n_f = 1 + r_f + r_f(r_f - c_f - 1)/\lambda_f$ (see (1.3.1)) we find that

(7.3.4) $$\lambda_f = 2\frac{(2r-k-1)(r-k-\lambda+x)}{b-2r+k}.$$

Denote the adjacency matrix of G_f by A_f, that is, from (7.1.18):

(7.3.5) $$A_f = C_1 + C_2 - I,$$

where the block adjacency matrix of C of $G_{\mathcal{B}}$ is

(7.3.6) $$C = \begin{pmatrix} C_1 & C_2 \\ C_2 & C_1 \end{pmatrix}.$$

The strongly regularity of G_f gives (from (2.3.5))

(7.3.7) $$A_f^2 = (r_f - \lambda_f)I + (c_f - \lambda_f)A_f + \lambda_f J.$$

We also have (from (4.1.3)(i))

(7.3.8 $$C^2 = (2r - \lambda)I + (c - 2\lambda)C + 2\lambda J - \begin{pmatrix} \bar{Y} & \bar{Y} \\ \bar{Y} & \bar{Y} \end{pmatrix},$$

where $\bar{Y}$ is the block intersection matrix of $\bar{D}$. From (7.3.3), (7.3.4), (7.3.5), (7.3.6), (7.3.7), and (7.3.8) we obtain

(7.3.9) $$\bar{Y} = (k - (2\lambda - \tfrac{1}{2}\lambda_f))I + (x - (2\lambda - \tfrac{1}{2}\lambda_f))A_f + (2\lambda - \tfrac{1}{2}\lambda_f)J$$

and therefore $\bar{D}$ is quasi-symmetric with intersection orders x and $y = 2\lambda - \frac{1}{2}\lambda_f$. Recall from Figure 16 that $|\beta_i^0 \cap \beta_j^{i_j}| = x$ for $1 \leq j \leq 2r - k - 1$, so that (4.1.13)(ii) gives

(7.3.10) $$x(2r - k - 1) + k = ck = k^2.$$

Hence

(7.3.11) $$x = \frac{k^2-k}{2r-k-1}.$$

But from (3.2.5) we have

(7.3.12) $$r = k^2 + \lambda - k\lambda,$$

so that

(7.3.13) $$x = \frac{k}{2k+1-2\lambda}.$$

The Stanton-Sprott equations (4.2.2) for quasi-symmetric $\tilde{D}$ become

(7.3.14) $$\begin{cases} m_x + m_y = b - 1, \\ xm_x + ym_y = k(r - 1), \\ x^2 m_x + y^2 m_y = k(r - 1) + k(k - 1)(\lambda - 1). \end{cases}$$

Since m_x = valence of $G_f = r_f = 2r - k - 1$ we find, using (7.3.11) and (7.3.14), that

(7.3.15) $$y = \frac{k(r-k)}{b-2r+k}.$$

Finally, from $b = 1 + r_f + r_f(r_f - c_f - 1)/\lambda_f$ (see (2.3.2)) we get

(7.3.16) $$b = r(r - k)/\lambda + r$$

so that (7.3.12) and (7.3.16) combine with (7.3.15) to yield

(7.3.17) $$y = \lambda/(k - \lambda).$$

If we express all $\bar{D}$ design parameters in terms of k, λ the theorem follows. □

The requirements on k and λ in (7.3.2) given by

(7.3.18) $$(k - \lambda)|\lambda, \quad (2k - 2\lambda + 1)|k, \quad \lambda|k^2(k - 1)$$

are indeed restrictive. We will discover the extent of this restriction after the next lemma.

(7.3.19) Lemma. All $k, \lambda \in Z^+$ that satisfy $(k - \lambda)|\lambda$ are given by $k = \mu(\beta + 1)$, $\lambda = \mu\beta$ where $\mu, \beta \in Z^+$.

Proof. Clearly $k - \lambda = \mu | \mu\beta$ so the prescribed k, λ are indeed solutions. Suppose $(k - \lambda)\delta = \lambda$ and hence $\delta k = \lambda(\delta + 1)$. Set $\mu = (k,\lambda)$ and $k = k_1\mu$, $\lambda = \mu\beta$. Then $\delta k_1 = \beta(\delta + 1)$ where $(\beta,k_1) = 1$. Since $\beta | \delta k_1$ and $(\beta,k_1) = 1$ we have $\beta | \delta$. Since $\delta | \beta(\delta + 1)$ and $(\delta,\delta + 1) = 1$ we have $\delta | \beta$. Hence $\delta = \beta$ and therefore $k_1 = \beta + 1 = \delta + 1$ proving the lemma.

(7.3.20) Corollary. The dodecahedral graph is the unique fusion strongly regular graph having $c = k$.

Proof. We need all positive integer solutions of (7.3.18). From (7.3.19) we have $k - \lambda = \mu \in Z^+$ so $(k - \mu)|k^2(k - 1)$ or $(k - \mu)|\mu^2(\mu - 1)$. Thus (7.3.18) is equivalent to

(7.3.21) $$(k - \mu)|\mu^2(\mu - 1), \quad (2\mu + 1)|k, \quad \mu|\lambda.$$

Letting $\lambda = \rho\mu$, we require

(7.3.22) $$(2\mu + 1)|\mu(\rho + 1), \quad \rho|\mu(\mu - 1).$$

Since $(2\mu + 1,\mu) = 1$ we need, equivalently, that $(2\mu + 1)|(\rho + 1)$, $\rho|\mu(\mu - 1)$. Let $(\rho + 1) = a(2\mu + 1)$, $\mu(\mu - 1) = b\rho$ for nonnegative integers a and b.

Then $\mu^2 - \mu(1 + 2ab) + b - ab = 0$. Hence $(2ab + 1)^2 - 4(b - ab) = t^2$ for some $t \in Z^+$. But this is possible if and only if

(7.3.23) $a = 1$ or $b = 0$.

For $b = 0$ we get

(7.3.24) $\mu = 1, \quad k = 3a, \quad \text{and} \quad \lambda = 3a - 1,$

whereas for $a = 1$ we get

(7.3.25) $k = 2\mu^2 + \mu, \quad \lambda = 2\mu^2, \quad \text{and} \quad \mu \equiv 1 \pmod 2, \quad \mu > 1.$

Consider now the spectrum of the adjacency matrix A_f of G_f. By (2.5.2) it is necessary that either $\lambda_f = c_f + 1$ or $(\lambda_f - c_f)^2 + 4(r_f - \lambda_f)$ is a square. For the case (7.3.24), $\lambda_f = c_f + 1$ becomes (by (7.3.3)) $a = 1$ and therefore $k = 3$, $\lambda = 2$. For (7.3.25), the case $\lambda_f = c_f + 1$ requires $k = 3$, $\lambda = 2$. In case $(\lambda_f - c_f)^2 + 4(r_f - \lambda_f)$ must be a square we find for (7.3.24) that $a^2 + 8a$ must be square so again $k = 3$ $\lambda = 2$. For (7.3.25) we find that $36\mu^2 - 12\mu - 15$ must be a square, which is impossible. Hence only $k = 3$, $\lambda = 2$ survive, i.e., the dodecahedral graph $G_{\bar{D}^2}$. □

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