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EIGENGRAPHS: CONSTRUCTING STRONGLY REGULAR GRAPHS WITH BLOCK DESIGNS

Donald M. Thompson

ABSTRACT. We define an eigengraph of a regular graph Λ as a subgraph Ω for which Ω and $\Lambda\backslash\Omega$ are both regular. It will be shown that Ω is closely related to an eigenvector for the adjacency matrix of Λ. The existence of a particular type of eigengraph in a strongly regular graph G imposes the structure of a balanced incomplete block design in G by means of a partition of the vertices of G into objects and blocks. We investigate the combinatorial and spectral implications for strongly regular graphs that possess this design structure, graphs which we call design constructible.

We give new constructions of some known strongly regular graphs by means of the design approach and in the process are able to determine some interesting subgraph structure for these graphs. We obtain results on the spectral properties of eigengraphs as well as combinatorial properties that they induce in the underlying graphs.

A particularly interesting design construction is given for a strongly regular graph on twenty-six vertices. The construction technique enables us to view the graph in terms of the faces and vertices of a regular dodecahedron as well as determine the automorphism group of the graph.

1. *Definitions and Notation.*

Definition 1.1. A *strongly regular graph* G, denoted $G = G(m,r,c,\lambda)$, is a regular graph of diameter 2 which is not complete or null, for which n is the number of vertices, r is the valence, c is the number of triangles on each edge, and λ is the number of paths of length 2 between any nonadjacent pair of vertices.

The facts mentioned in this section may be found in Hestenes and Higman (1971) or Cameron and Van Lint (1975). Simple counting arguments yield

$$n = 1 + r + r(r - c - 1)/\lambda \ . \tag{1.2}$$

We will write A for the adjacency matrix of G, having its rows and columns indexed by V(G), the vertex set of G, so that

$$(1.3) \qquad A_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly

$$(1.4) \qquad AJ = JA = rJ ,$$

where J is the all 1 matrix. Moreover,

$$(1.5) \qquad A^2 = (r-\lambda)I + (c - \lambda)A + \lambda J .$$

For any matrix M having distinct eigenvalues $\rho_1 > \rho_2 > \ldots > \rho_m$ with respective multiplicities $z_1, z_2, \ldots, z_m$, we will write

$$\operatorname{spec}(M) = \begin{bmatrix} \rho_1 & \rho_2 & \ldots & \rho_m \\ z_1 & z_2 & & z_m \end{bmatrix}$$

for the spectrum of M. In particular, it was shown in Hestenes and Higman (1971) that

$$(1.6) \qquad \operatorname{spec}(A) = \begin{bmatrix} r & \rho_2 & \rho_3 \\ z_1 = 1 & z_2 & z_3 \end{bmatrix},$$

where

$$(1.7) \qquad \begin{Bmatrix} \rho_2 \\ \rho_3 \end{Bmatrix} = \tfrac{1}{2}(c - \lambda \pm \sqrt{(\lambda - c)^2 + 4(r - \lambda)})$$

and where

$$(1.8) \qquad \begin{cases} z_2 = \dfrac{\rho_3(n-1) + r}{\rho_3 - \rho_2}, \\ z_3 = n - z_2 - 1 . \end{cases}$$

If $G = G(n,r,c,\lambda)$ is strongly regular then so is its complement

$G' = G'(n',r',c',\lambda')$, (which is possibly disconnected), the graph whose adjacency matrix is $J - A - I$. We have

$$\begin{aligned} n' &= n, \\ r' &= n - r - 1, \\ c' &= r' - r + \lambda - 1, \\ \lambda' &= r' - r + c + 1. \end{aligned}$$

For any graph G and $w \in V(G)$, the vertex set of G, we will write

$$(1.9) \qquad \begin{cases} \Gamma_G(w) = \{u \in V(G) : u \sim w\} , \\ \Delta_G(w) = \{u \in V(G) : u \not\sim w, u \neq w\}. \end{cases}$$

In conjunction with the structure of strongly regular graphs we will employ the idea of another incidence structure - a block design.

Definition 1.10. A *block design* $D = D(v,b,r,k,\lambda)$ is an arrangement of v distinct objects into b sets of equal size k called blocks in such a way that each object appears in r blocks, and every pair of distinct objects appears together in λ blocks.

The well known equations

$$(1.11) \qquad \begin{cases} bk = vr, \\ \lambda(v-1) = r(k - 1), \end{cases}$$

may be found in Hall (1967). A *complete* block design is formed by taking all subsets of size k from a set of size v to form $b = \binom{v}{k}$ blocks. We will denote the complete block design by K_k^v, having parameters $(v, \binom{v}{k}, \binom{v-1}{k-1}, k, \binom{v-2}{k-2})$.

Objects in a block design D will be denoted $\{o_1,\ldots,o_v\}$ or, when convenient, simply $\{1,2,\ldots,v\}$. Blocks will be written as $\beta = (o_1,o_2,\ldots,o_k)$ and we will alternately view β as a set, for intersection purposes, or as a vertex, for graph purposes. The object-block incidence matrix B of D is that $v \times b$ matrix for which

$$B_{ij} = \begin{cases} 1 & \text{if } o_i \in \beta_j , \\ 0 & \text{otherwise .} \end{cases}$$

Hence, since $(BB^T)_{ij}$ counts the number of blocks containing o_i and o_j, we have

(1.12) $$BB^T = (r - \lambda)I + \lambda J ,$$

and

(1.13) $$\mathrm{spec}(BB^T) = \begin{Bmatrix} rk & r-\lambda \\ 1 & v-1 \end{Bmatrix}.$$

Since Fisher's inequality (see Hall (1967)) says

(1.14) $$b \ge v, \qquad (r \ge k)$$

then

(1.15) $$\mathrm{spec}(B^TB) = \begin{Bmatrix} rk & r-\lambda & 0 \\ 1 & v-1 & b-v \end{Bmatrix}.$$

We may interpret $(B^TB)_{ij}$ as the number of objects common to blocks β_i and β_j. From Definition 1.10 and this remark we see that

(1.16) $$\begin{cases} B^TJ = kJ, \\ BJ = rJ, \\ (B^TB)J = J(B^TB) = rkJ \end{cases},$$

for choices of J of appropriate sizes. Hence, by (1.12),

(1.17) $$(B^TB)^2 = (r-\lambda)B^TB + k^2\lambda J .$$

$PG(n,q)$ will denote the projective geometry of dimension n over $GF(q)$. This geometry may be viewed as a block design where the geometry's points serve as objects and the geometry's lines serve as blocks (see Hall (1967)).

In order to combine block designs with graphs we need the next definition.

Definition 1.18. An *independent set* in a graph is a set of mutually nonadjacent vertices.

2. *Design Constructibility.*

Definition 2.1. Call a strongly regular graph $G = G(n,r,c,\lambda)$ *design constructible* (d.c.) if the vertices of G can be partitioned into two sets V and $\mathcal{B}$ such that

(i) V is an independent set,

(ii) $|\{o_i \in V : o_i \sim \beta\}|$ is independent of the choice of $\beta \in \mathcal{B}$.

Definition 2.2. Call a set V of vertices which satisfies Definition 2.1 (i) and (ii) an *object set.*

THEOREM 2.3. *Let* $G = G(n,r,c,\lambda)$ *be a design constructible strongly regular graph. Then*

(i) V *and* $\mathcal{B}$ *form a block design* D;

(ii) the parameters of D *are given by* $v = r(-\rho_3 - 1)/\lambda + 1$, $b = n-v$, $r = r$, $k = -\rho_3$, $\lambda = \lambda$. *Here* ρ_3 *is the unique negative eigenvalue of the adjacency matrix of* G.

Proof. Write $V = \{o_1,o_2,\dots,o_v\}$, $\mathcal{B} = \{\beta_1,\dots,\beta_b\}$, where $v = |V|$, $b = n - v = |\mathcal{B}|$. V is an object set, and we shall call the vertices in $\mathcal{B}$ blocks. Define the following incidence relation on $V \cup \mathcal{B}$: $o_i \in \beta_j$ if and only if $o_i \sim \beta_j$ in G. By definition $o_i \not\sim o_j$, for all i, j. Thus $\Gamma_G(o_i) \subseteq \mathcal{B}$, $1 \le i \le v$, so that each object is contained in (is adjacent to) r blocks. Since Definition 2.1(ii) is in force, each block contains $k \equiv |\{o_i \in V: o_i \sim \beta_j\}|$ objects. Finally, since nonadjacent vertices of G have λ paths between them, any pair of objects is contained in λ blocks. Thus V and $\mathcal{B}$ define a block design, establishing (i). For (ii) we use (1.11) which says that $\lambda(v-1) = r(k-1)$ and $(n-v)k = vr$. Eliminating v from these equations yields

(2.4) $$rk^2 + k(r(r-1) - \lambda(n-1)) + r(\lambda-r) = 0.$$

Applying (1.2) to (2.4) gives

(2.5) $$k^2 + k(c-\lambda) - (r-\lambda) = 0.$$

Hence $k = \frac{1}{2}\left(\lambda-c \pm \sqrt{(\lambda-c)^2 + 4(r-\lambda)}\right)$, and $k > 0$ demands we take the plus sign so that from (1.7) we have

(2.6) $k = -\rho_3$.

The stated form for v follows from (1.11) and (2.6) □

In case G is d.c. by the design D we will say that D *affords* G and that (G,D) is an *affordable pair*. $G_{\mathcal{B}}$ will denote $G|\mathcal{B}$ and, for $\beta \in \mathcal{B}$, $\Gamma_{G_{\mathcal{B}}}(\beta)$ will denote the set of $r-k$ blocks adjacent to β, $\Delta_{G_{\mathcal{B}}}(\beta)$ denoting the $b-(r-k)-1$ blocks not adjacent to β.

Regarding v, a result of Haemers (1978) is worth mentioning.

THEOREM 2.7. *Let* H *be a regular graph on* n *vertices whose adjacency matrix has eigenvalues* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, *(*$\lambda_1$ *= valence). Then any independent set in* H *contains at most* $n(-\lambda_n)/(\lambda_1-\lambda_n)$ *vertices.*

Applying Theorem 2.7 to a d.c. $G = G(n,r,c,\lambda)$ gives a bound of $nk/(r+k)$ on the number of independent vertices. From (1.11) and Theorem 2.3 the bound becomes

$$\frac{(b+v)k}{r+k} = \frac{v(r+k)}{r+k} = v,$$

so that in our case an object set has maximal order.

Before generalizing the idea of design constructibility, we cite an unusual instance where a strongly regular graph G is not d.c. while its complement G' is d.c.

THEOREM 2.8. *No strongly regular* $G = G(27,10,1,5)$ *is d.c., whereas there is a strongly regular* $G' = G'(27,16,10,8)$ *which is d.c.*

Proof. In order that any $G = G(27,10,1,5)$ be d.c. we require (using Theorem 2.3) a design D with parameters (9,18,10,5,5). Such a design exists (see Hall (1967)). Set $V = \{1,2,3,\ldots,9\}$. Let $\beta = (1,2,3,4,5)$ be a typical block of any D having the stated parameters. Then $\Gamma_G(\beta) = \{\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,1,2,3,4,5\}$ must be as in Figure 1.

Since $c = 1$ then $\{\beta_i \cap \beta\} = \{i\}$ and thus $\{\beta_i \cap (V-\beta)\} = \{6,7,8,9\}$, $1 \leq i \leq 5$. From $\lambda = 5$ we see that $\Gamma_{G_{\mathcal{B}}}(\beta_i) \cap \Gamma_{G_{\mathcal{B}}}(\beta_j) = \beta$, $1 \leq i, j \leq 5$, $i \neq j$. Hence $\Gamma(\beta_i)$, $i = 1,\ldots,5$, contributes a total of twenty new blocks beyond β, β_1, β_2, β_3, β_4, β_5. But $b = 18$, a contradiction. Hence G is not d.c.

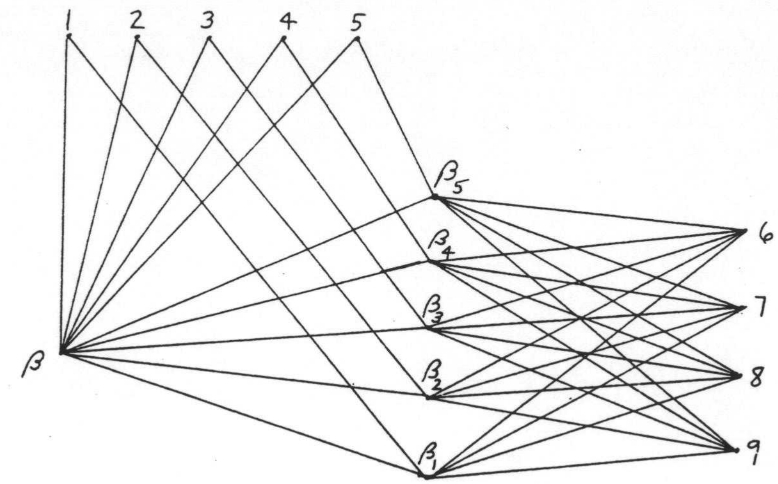

Figure 1. $\Gamma_G(\beta)$.

The G' we speak of is the Schäfli graph (see Cameron et al. (1976)), which is represented as follows: the vertices of G' are the vectors

$$\{\vec{e}_i + \vec{e}_j : 1 \leq i, j \leq 6, i \neq j\}$$
$$\cup \left\{\tfrac{1}{2}\sum_{k=1}^{8} e_k - \vec{e}_i - \vec{e}_7,\ \tfrac{1}{2}\sum_{k=1}^{8} \vec{e}_k - \vec{e}_i - \vec{e}_8 : 1 \leq i \leq 6\right\},$$

where $\vec{e}_m$ is the standard basis vector of R^8 which has a 1 in position m, 0 elsewhere. Two vertices are adjacent if and only if their dot product is 1. It is easily checked that $V = \{\vec{e}_1 + \vec{e}_2, \vec{e}_3 + \vec{e}_4, \vec{e}_5 + \vec{e}_6\}$ is an independent set for which all vertices not in V are adjacent to precisely two vertices in V. Hence G is d.c. via $D' = D'(3,24,16,2,8)$ which happens to be eight identical copies of K_2^3. □

3. *Eigengraphs.*

We have already seen the close connection between the design parameter k and spec(A). There is an algebraic interpretation of this connection which generalizes to other types of subgraphs (besides the object subgraph) of a strongly regular graph.

Let Λ be any regular graph of valence r on n vertices whose adjacency matrix A_Λ has integer eigenvalues. If Ω is a subgraph of Λ denote χ_Ω to be the characteristic vector of Ω having

$$(3.1) \qquad (\chi_\Omega)_i = \begin{cases} 1 & \text{if vertex } i \text{ is in } \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.2. Call a subgraph Ω of the regular graph Λ an *eigengraph* of Λ if

(i) Ω is regular of some valence ξ, and

(ii) every vertex in $\Lambda\backslash\Omega$ is adjacent to the same number η of vertices in Ω.

THEOREM 3.3. *The graph* Ω *is an eigengraph of a regular (r-valent) graph* Λ *if and only if*

(i) $A_\Lambda\chi_\Omega = r\chi_\Omega$, *or*

(ii) $\xi - \eta$ *is an eigenvalue of* A_Λ *with*

$$\chi_\Omega - \frac{\eta}{r-(\xi-\eta)}\chi_\Lambda$$

a corresponding eigenvector.

Proof. Note first that Definition 3.2 is equivalent to saying

$$(3.4) \qquad A_\Lambda\chi_\Omega = (\xi-\eta)\chi_\Omega + \chi_\Lambda ,$$

since $(A_\Lambda\chi_\Omega)_i$ counts the number of vertices in Ω that are adjacent to a Λ - vertex i. For $\xi = r$, $\eta = 0$ we get that Theorem 3.3 (i) follows from (3.4) and conversely.

That (3.4) and Theorem 3.3 (ii) are equivalent (for $\xi-\eta \neq r$) is a matter of using $A_\Lambda\chi_\Lambda = r\chi_\Lambda$ and some matrix multiplication. □

If $\rho \neq r$ is an eigenvalue of A_Λ corresponding to the eigengraph Ω, we will say that Ω is *type* (ρ).

An algebraic interpretation of design constructibility is now possible. For $\Lambda = G$, a strongly regular graph (with integer eigenvalues) we see that an object set is merely an eigengraph of type $(-k)$

having $\xi = 0$, $\eta = k$. Thus design constructibility is equivalent to a condition on the structure of the $(-k)$ eigenspace of A.

From now on we shall write the spectrum of A (the adjacency matrix of a strongly regular $G = G(n,r,c,\lambda)$) as

$$(3.5) \qquad \operatorname{spec}(A) = \begin{Bmatrix} r & s-k & -k \\ 1 & z_2 & z_3 \end{Bmatrix},$$

where

$$(3.6) \qquad s = \sqrt{(\lambda-c)^2 + 4(r-\lambda)} = 2k - (\lambda-c) .$$

Although this paper devotes its attention to object set eigengraphs, it is reasonable to consider other type $(-k)$ as well as type $(s-k)$ eigengraphs. In the process we make another observation:

COROLLARY 3.7. *If* Ω *is a type* (ρ) *eigengraph of* Λ *then* $\Lambda\backslash\Omega$ *is an eigengraph of type* (ρ).

Proof. Note that $\chi_{\Lambda\backslash\Omega} = \chi_\Lambda - \chi_\Omega$. Say that $A_\Lambda\chi_\Omega = (\xi-\eta)\chi_\Omega+\eta\chi_\Lambda = \rho\chi_\Omega+\eta\chi_\Lambda$, then $A_\Lambda(\chi_\Lambda-\chi_\Omega) = \rho(\chi_\Lambda-\chi_\Omega)+(r-\xi)\chi_\Lambda$ and the corollary follows. □

Let Ω_ρ denote a type (ρ) eigengraph of a regular graph Λ of valence r. Denote by $|\Omega_\rho|$ the number of vertices in Ω_ρ. The next theorem allows us to compute $|\Omega_\rho|$ as well as $|\Omega_{\rho_i} \cap \Omega_{\rho_j}|$:

THEOREM 3.8. *Suppose* Λ *is a regular graph of valence* r *on* n *vertices. Let the adjacency matrix* A_Λ *of* Λ *have distinct integer eigenvalues* $r > \rho_2 > \rho_3 > \ldots > \rho_m$. *Let the valence of a type* (ρ_i) *eigengraph* Ω_{ρ_i} *be denoted* ξ_i, $2 \le i \le m$. *Then*

(i) $|\Omega_{\rho_i}| = \dfrac{n\eta_i}{r-\rho_i}$, $2 \le i \le m$, *where* $\eta_i = \xi_i - \rho_i$;

(ii) $|\Omega_{\rho_i} \cap \Omega_{\rho_j}| = |\Omega_{\rho_i}|\,|\Omega_{\rho_j}|/n$, $2 \le i, j \le m$, $i \neq j$.

Proof. A_Λ is symmetric, hence distinct eigenvalues afford orthogonal eigenvectors. Since χ_Λ is an eigenvector for r then

$$\left(\chi_{\Omega_{\rho_i}} - \frac{n_i}{r-\rho_i}\chi_\Lambda, \chi_\Lambda\right) = 0, \quad 2 \le i \le m,$$

which reduces to (i) since $(\chi_{\Omega_{\rho_i}}, \chi_\Lambda) = |\Omega_{\rho_i}|$, $(\chi_\Lambda,\chi_\Lambda) = n$. For (ii) we have

$$\left(\chi_{\Omega_{\rho_i}} - \frac{n_i}{r-\rho_i}\chi_\Lambda, \chi_{\Omega_{\rho_j}} - \frac{n_j}{r-\rho_j}\chi_\Lambda\right) = 0 .$$

Hence, by (i):

$$|\Omega_{\rho_i} \cap \Omega_{\rho_j}| = (\chi_{\Omega_{\rho_i}}, \chi_{\Omega_{\rho_j}})$$

$$= \frac{n n_i n_j}{(r-\rho_i)(r-\rho_j)} = |\Omega_{\rho_i}|\,|\Omega_{\rho_j}|/n. \qquad \square$$

We now cite a few examples of type $(s-k)$ eigengraphs which are themselves strongly regular.

Example 3.9. Note that the Hoffman-Singleton graph $G = G(50,7,0,1)$ has spectrum $\begin{Bmatrix} 7 & 2 & -3 \\ 1 & 28 & 21 \end{Bmatrix}$. Any type (2) eigengraph Ω_2 has (via Theorem 3.8) 10η vertices with valence $\xi = 2 + \eta$. For Ω_2 to be strongly regular we would need 0 triangles on each edge and one path of length two between nonadjacent vertex pairs. For $c = 0$, $\lambda = 1$ there are but four possible graph sizes (see Higman (1964)), namely $n = 5$, giving the pentagon; $n = 10$, giving the Petersen graph; $n = 50$, giving the Hoffman-Singleton graph; and $n = 3250$, a graph whose existence is unknown; comprising the so-called Moore graphs. Hence we must choose $\eta = 1$ and get $\Omega_2 = G(10,3,0,1)$, the Petersen graph, (an embedding property which is well known). Applying the same procedure to the Petersen graph we verify that it contains the pentagon as a (type $(s-k)$) eigengraph. Having obtained this sequence of embedded eigengraphs, we naturally ask if the largest Moore graph $G = G(3250,57,0,1)$ could have the Hoffman-Singleton graph as an eigengraph of either type? After computing the spectrum we find that, unfortunately, this is impossible.

Any type $(s-k)$ eigengraph in $G = G(3250,57,0,1)$ would have 65η vertices and valence $\eta + 7$, while a type $(-k)$ eigengraph would have 50η vertices and valence $\eta - 8$.

Example 3.10. The Petersen graph can be partitioned into two type $(s-k)$ eigengraphs, namely two pentagons. The Hoffman-Singleton graph can be partitioned into five disjoint type $(s-k)$ eigengraphs, namely five Petersen graphs. The Higman-Sims graph $G = G(100,22,0,6)$ (see Gewirtz (1969)) has been shown by Sims (1969) to contain two disjoint Hoffman-Singleton graphs, indeed two type $(-k)$ eigengraphs.

4. *Spectral and Matrix Properties.*

The presence of an object set of size v in a strongly regular graph induces many interesting spectral relationships between the adjacency matrix A of the graph, the block intersection matrix $Y = B^TB$ of the design, and the $b \times b$ submatrix C of A describing block adjacencies. In this section we determine $\mathrm{spec}(C)$ and develop several matrix relationships between Y and C.

Assume that $G = G(n,r,c,\lambda)$ is afforded by $D = D(v,b,r,k,\lambda)$ with design parameters as described in Section 2. Index the vertices of G so that the first v vertices $\{o_1,\dots,o_v\}$ correspond to the objects and the last b vertices $\{\beta_1,\dots,\beta_b\}$ correspond to the blocks of D. Let B be the $v \times b$ object-block incidence matrix of D (as in (2.4)) and denote by C the $b \times b$ matrix for which $C_{ij} = 1$ if $\beta_i \sim \beta_j$, 0 otherwise. We write A as

$$A = \begin{bmatrix} 0 & B \\ B^T & C \end{bmatrix}. \tag{4.1}$$

THEOREM 4.2. *The spectrum of* C *is*

$$\mathrm{spec}(C) = \begin{bmatrix} r-k & s-k & s-2k & -k \\ 1 & z_2-(v-1) & v-1 & z_3-v \end{bmatrix},$$

where z_2, z_3 *are as in (1.8).*

We need a lemma before beginning the proof.

LEMMA 4.3.

(i) $C^2 = (r-\lambda)I + (c-\lambda)C + \lambda J - Y,$

and

(ii) $BC = (c-\lambda)B + \lambda J_{v\times b}.$

Proof. From (1.5) and (4.1) we get

$$\begin{bmatrix} BB^T & BC \\ CB^T & C^2+Y \end{bmatrix} = (r-\lambda)\begin{bmatrix} I_{v\times v} & 0 \\ 0 & I_{b\times b} \end{bmatrix} + (c-\lambda)\begin{bmatrix} 0 & B \\ B^T & C \end{bmatrix} + \lambda J \ .$$

The lemma follows. □

Proof of Theorem 4.2. We multiply Lemma 4.3 (ii) on the left by B^T to obtain, via (1.16),

$$(4.4) \qquad YC = (c-\lambda)Y + k\lambda J \ .$$

Note also that

$$(4.5) \qquad CJ = JC = (r-k)J \ .$$

We eliminate Y from (4.4) and Lemma 4.3 to obtain

$$(4.6) \qquad \begin{aligned} &C^3 + 2(\lambda-c)C^2 + ((\lambda-c)^2 - (r-\lambda))C \\ &\quad - (r-\lambda)(\lambda-c)I = \lambda J(r-(2k-\lambda+c)) \ . \end{aligned}$$

Multiply (4.6) by C, then eliminate J from the resulting equation via (4.6) to obtain

$$(4.7) \qquad \begin{aligned} &C^4 - C^3(2(c-\lambda) + r - k) \\ &\quad - C^2((r-\lambda) - (\lambda-c)^2 - 2(r-k)(c-\lambda)) \\ &\quad - C((r-\lambda)(\lambda-c) - (r-k)(r-\lambda-(\lambda-c)^2)) \\ &\quad + (r-k)(\lambda-c)(r-\lambda)I = 0 \ . \end{aligned}$$

Since $r-\lambda = k(s-k)$, $c-\lambda = s - 2k$ (by (2.5) and (3.6)), we may rewrite (4.7) as

$$(4.8) \qquad (C - (r-k)I)(c - (-k)I)(C - (s-k))\ (C - (s-2k)I) = 0 \ .$$

Thus we have

$$(4.9) \qquad \mathrm{spec}(C) = \begin{bmatrix} r-k & s-k & s-2k & -k \\ w_1 & w_2 & w_3 & w_4 \end{bmatrix}$$

for appropriate (positive integer) multiplicities w_1, w_2, w_3, w_4. Note that

$$(4.10) \qquad \mathrm{tr}(C^e) = w_1(r-k)^e + w_2(s-k)^e + w_3(s-2k)^e + w_4(-k)^e$$

and that C^e_{ii} counts the number of closed paths of length e starting and ending at β_i. For $e = 0, 1, 2$ the counting is simple. For $e = 3$ we get

$$(4.11) \qquad C^3_{ii} = c(r - 2k), \quad 1 \le i \le b,$$

which follows from inspecting the diagonal elements of (4.6).

Consequently

$$(4.12) \qquad \begin{cases} \mathrm{tr}(C^0) = b \ , \\ \mathrm{tr}(C) \ = 0 \ , \\ \mathrm{tr}(C^2) = b(r-k) \ , \\ \mathrm{tr}(C^3) = bc(r-2k) \ . \end{cases}$$

Combining (4.12) with (4.10) and using (1.8), Theorem 2.3 (ii), and (1.11) gives Theorem 4.2. □

THEOREM 4.13. $G(n,r,c,\lambda)$ *is afforded by* $D = D(v,b,r,k,\lambda)$ *if and only if*

(i) (1.12) *holds,*

(ii) $\Gamma_{G_B}(\beta)$ *contains* c *repetitions of the* k *objects in* β *and* λ *repetitions of the remaining* $v-k$ *objects of* D, *for all* $\beta \in \mathcal{B}$, *and*

(iii) *the number of blocks adjacent to both* β_i *and* β_j *is*

(a) $r-k$ *if* $i = j$,

(b) $c-Y_{ij}$ *if* $\beta_i \sim \beta_j$,

(c) $\lambda-Y_{ij}$ *if* $\beta_i \not\sim \beta_j$, *for all* i,j.

Proof. For (ii) notice that $(BC)_{ij}$ counts the number of repetitions of object o_i in $\Gamma_{G_B}(\beta_j)$; hence (ii) is equivalent to Lemma 4.3 (ii). Clearly (iii) is equivalent to Lemma 4.3 (i). Thus the theorem says that (G,D) is an affordable pair if and only if Lemma 4.3 and (1.12) hold. Certainly Lemma 4.3 and (1.12) are necessary for design constructibility.

For sufficiency observe that we need only establish (1.4) and (1.5) when A is written as in (4.1). But Lemma 4.3 and (1.12) give us (1.5), while the row (column) sum of r for A follows directly from Theorem 4.13 (ii), (iii), and (1.12). □

A situation that utilizes Theorem 4.13, in the negative sense is

COROLLARY 4.14. *No graph* $G = G(16,5,0,2)$ *is design constructible.*

Proof. In order for a design D to afford $G(16,5,0,2)$ we require (via Theorem 2.3) the design parameters $(6,10,5,3,2)$. Hall (1967) established the uniqueness of such a design. We will represent the blocks of D as

(4.15) $\{(1,2,3),(1,2,5),(1,4,6),(1,5,6),(1,3,4),$
$(3,4,5),(2,3,6),(2,4,5),(3,5,6),(2,4,6)\}.$

Note that every pair of distinct blocks intersects in one or two objects. Hence it is impossible to satisfy Theorem 4.13 (ii) on the $r-k = 2$ blocks adjacent to a given block. Thus $G = G(16,5,0,2)$ (although it has been constructed by other means, Biggs (1971)) is not design constructible. □

5. *Design Constructions of Some Known Graphs.*

There is an abundance of examples of known strongly regular graphs G that can be constructed by finding an appropriate design D affording G. Indeed, any strongly regular graph having a transitive automorphism group and possessing an object set is design constructible, the constancy of the block size being guaranteed by vertex transitivity. In the interest of symmetry, we present some of the more "coherent" constructions, that is, those for which the block adjacency rule is independent of block choice. The following result has also been noted by Seidel (1976). We include its proof in order to detail an important example of design constructibility.

THEOREM 5.1. *The Hoffman-Singleton graph is afforded by* $PG(3,2)$.

Proof. From Theorem 2.3 we need a design $D = D(15,35,7,3,1)$. There are eighty nonisomorphic designs with these parameters (see Cole, White, and Cummings (1925)). We will first establish the reasons for singling out $PG(3,2)$. For a block β, $\Gamma_{G_B}(\beta)$ must consist of four mutually non-intersecting blocks $(\lambda = 1)$, which are all disjoint from β $(c = 0)$. Writing $\Gamma_{G_B}(\beta) = \{\beta_1,\beta_2,\beta_3,\beta_4\}$ we require then that $\{\beta,\beta_1,\beta_2,\beta_3,\beta_4\}$ constitute a complete replication of the fifteen objects, i.e., $\{\beta \cup \Gamma_{G_B}(\beta)\}$ is a parallel class. Having chosen $\Gamma_{G_B}(\beta)$, we then look at $\beta_i (i = 1,2,3,4)$. Each $\{\beta_i \cup \Gamma_{G_B}(\beta_i)\}$, $1 \le i \le 4$, must be a parallel class, and since $\lambda = 1$, $c = 0$ we have $\Gamma_{G_B}(\beta_i) \cap \Gamma_{G_B}(\beta_j) = \beta$, $1 \le i, j \le 4$, $i \ne j$ (see Figure 2).

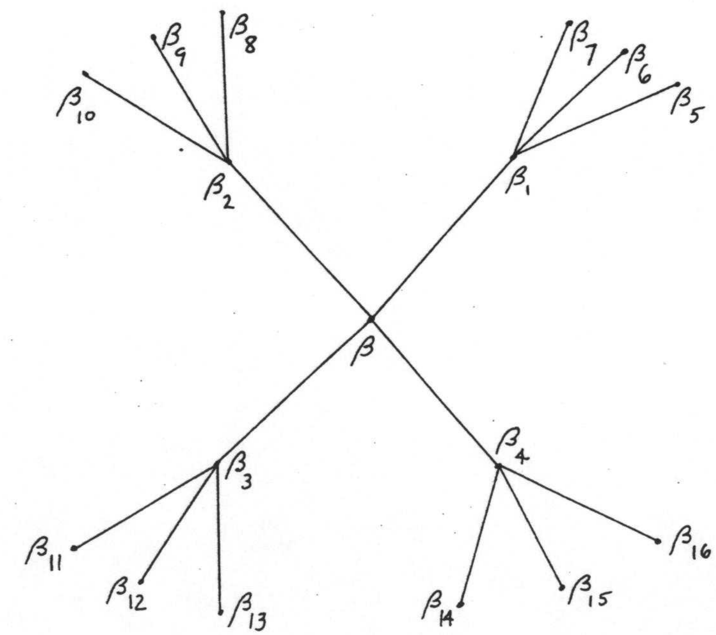

Figure 2. $\Gamma_{G_B}(\beta)$, $\Gamma_{G_B}(\beta_i)$ $(1 \le i \le 4)$.

Consider the seven blocks $\{\beta_1^j, \beta_2^j, \ldots, \beta_7^j\}$ containing the common object j, $j = 1,2,\ldots,15$. Since $\lambda = 1$, $\Gamma_{G_B}(\beta_i^j) \cap \Gamma_{G_B}(\beta_\ell^j) = \emptyset$ for $i, \ell = 1,\ldots,7$, $i \neq \ell$. Thus the design must be resolvable, i.e., we must be able to partition its thirty-five blocks into seven parallel classes.

The resolvability requirement is precisely what led to the *Kirkman Schoolgirl problem*, posed and solved by Kirkman (1847). A schoolteacher takes her class of fifteen girls on a daily walk. The girls are arranged in five rows of three each, so that each girl has two companions. The problem is to arrange the girls so that for seven consecutive days no girl walks with one of her companions in a triplet more than once.

The Kirkman design thus obtained is isomorphic to the design obtained by taking as objects the fifteen points of PG(3,2) and as blocks the thirty-five lines of PG(3,2). Not only is PG(3,2) resolvable but it has the largest automorphism group $(PSL(4,2) \cong A_8)$ among the eighty designs with the same parameters (see Cole et al. (1925)).

The following resolution of PG(3,2) serves to give a new construction for the Hoffman-Singleton graph.

Parallel Class					
1	{(1,2,3),	(6,8,14),	(5,9,12),	(4,11,15),	(7,10,13)}
2	{(1,4,5),	(2,8,10),	(6,9,15),	(3,13,14),	(7,11,12)}
3	{(1,6,7),	(2,9,11),	(5,8,13),	(4,10,14),	(3,12,15)}
4	{(1,8,9),	(3,4,7),	(2,13,15),	(6,10,12),	(5,11,14)}
5	{(1,10,11),	(2,12,14),	(3,5,6),	(4,9,13),	(7,8,15)}
6	{(1,12,13),	(2,4,6),	(3,8,11),	(5,10,15),	(7,9,14)}
7	{(1,14,15),	(2,5,7),	(4,8,12),	(6,11,13),	(3,9,10)} .

Table 5.2.

The geometry PG(3,2) has another property that can be exploited, namely, given two disjoint blocks β, $\bar\beta$, there exist exactly two parallel classes containing β and $\bar\beta$; and these classes have only β, $\bar\beta$ in common. For example, with (1,2,3) and (5,9,12) we get {(4,11,15), (6,8,14), (7,10,13)} and {(4,10,14), (7,8,15), (6,11,13)}.

We now present the rule describing block adjacency in G_B.

(5.3). (i) Any block of the form (1,i,j) is connected to the four blocks in the same parallel class (in Table 5.2) as (1,i,j).

(ii) Any block $\bar\beta$ of the form $\bar\beta = (i,j,k)$, $i, j, k \neq 1$, is connected to the block β which contains 1 and is in the same parallel class (in Table 5.2) as $\bar\beta$, and to the uniquely determined three further blocks which form another parallel class on β and $\bar\beta$.

Thus for example we have the situation of Figure 3 for the blocks at distances one and two from (1,2,3) in G_B.

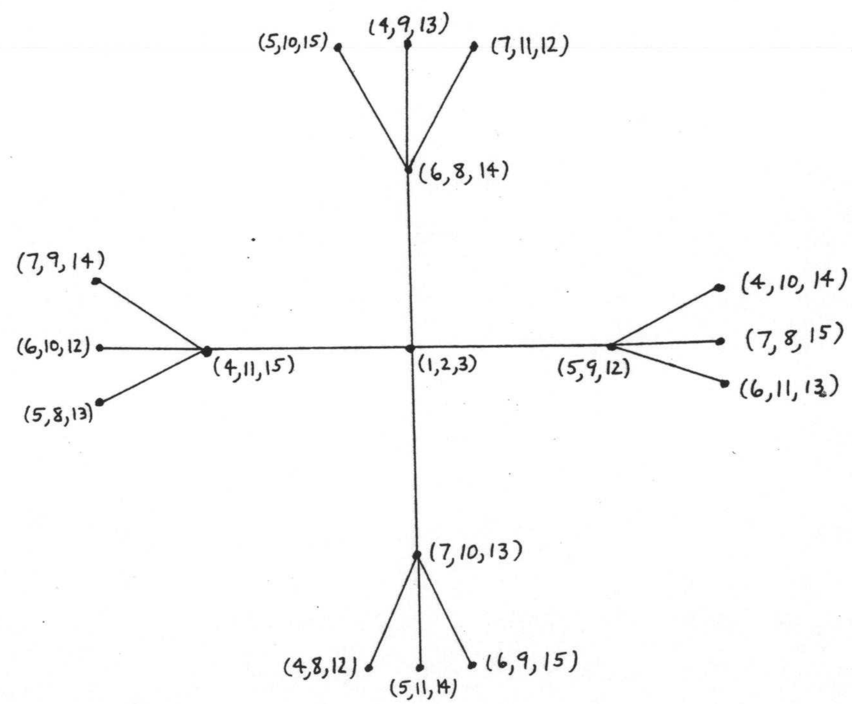

Figure 3. $\{\beta \in PG(3,2): \partial_{G_B}(\beta,(1,2,3)) \leq 2\}$.

It is routine to check that (5.3) defines the correct block adjacency rule, thus providing a design construction for the Hoffman-Singleton graph. (For the full $G_{\mathcal{B}}$ see Figure 4.)

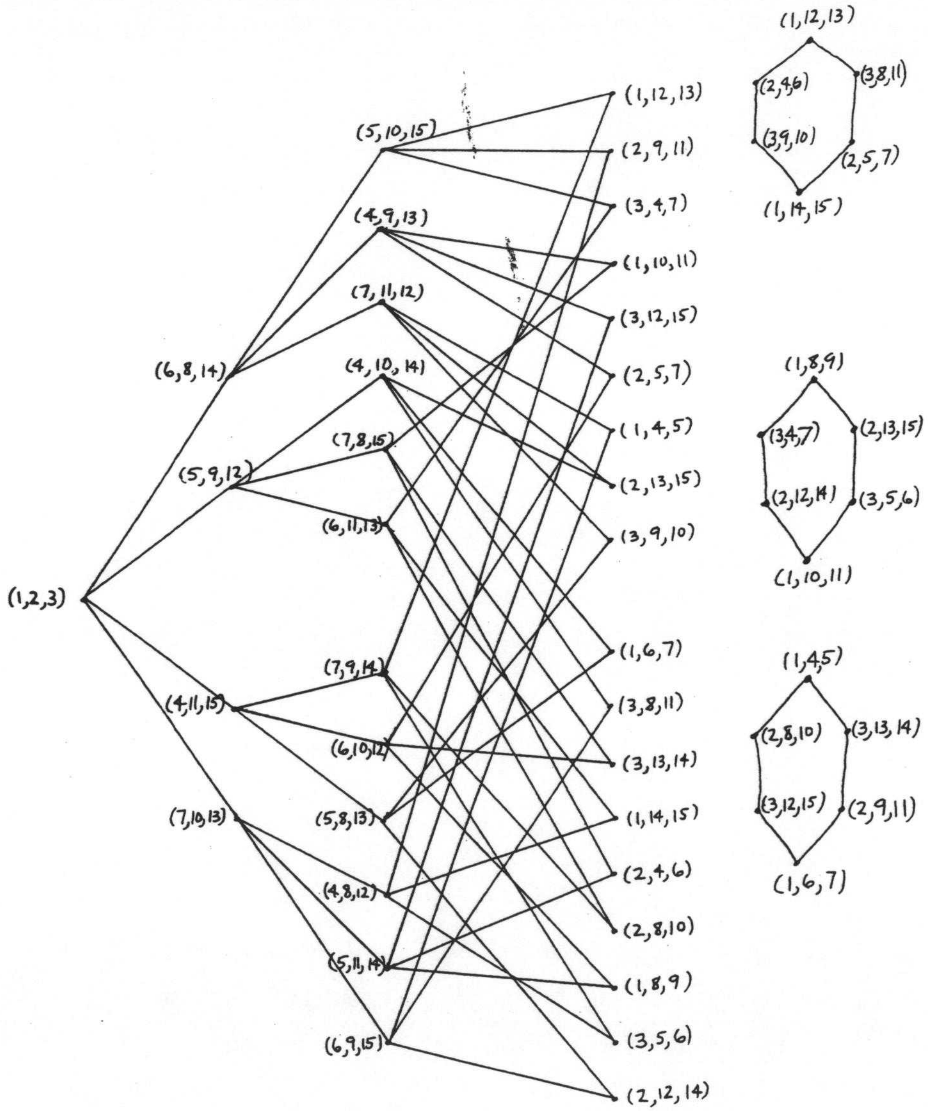

Figure 4. The Block Graph of the Hoffman-Singleton Graph.

A design affording the next Moore graph $G = G(3250,57,0,1)$ would have to have parameters $(400,2850,57,8,1)$, which are precisely the parameters of $PG(3,7)$. We conjecture that a resolution approach similar to the one employed in Theorem 5.1 should be taken. The difficulty here, of course, is the size of the design.

Definition 5.4.

(i) In a design $D = D(v,b,r,k,\lambda)$ distinguish a block β and set $m_i = |\{\bar{\beta} \in \mathcal{B}\backslash\beta : |\bar{\beta} \cap \beta| = i\}|$, $0 \le i \le k$. The set $\{m_0, m_1, \ldots, m_k\}$ will be called the *intersection distribution* of block β.

(ii) A design D is called *quasi-symmetric* if every block has the same intersection distribution, consisting of only 2 nonzero values.

As observed by Cameron and Van Lint (1975), quasi-symmetric designs afford strongly regular graphs when the blocks comprise all of the vertices. Here two blocks are connected if and only if they intersect in a certain order. This is not the only use that can be made of such designs for constructing strongly regular graphs. We present here two known graphs, each constructed in a new way via quasi-symmetric designs.

A well known family of strongly regular graphs is obtained from the projective geometries $PG(3,q)$, wherein the vertices of the graph are the $n = (q^2+1)(q^2+q+1)$ lines of the geometry and two lines are adjacent if and only if they intersect (see Biggs (1971)). The parameters of the resulting graph are

$$((q^2+1)(q^2+q+1),\ q(q+1)^2,\ 2q^2+q-1,\ (q+1)^2)\ .$$

We present the design construction of $PG(3,2)$.

THEOREM 5.5. *The graph* $G = G(35,18,9,9)$ *is afforded by three copies* D_1, D_2, D_3 *of* K_3^5 *by means of the following rule: if a block of copy* μ *is denoted by* β^μ *then*

$$\Gamma_{G_{\mathcal{B}}}(\beta^\mu) = \{\bar{\beta}^\mu \in D_\mu : |\beta^\mu \cap \bar{\beta}^\mu| = 1\}$$
$$\bigcup \{\bar{\beta}^\sigma \in D_\sigma : \sigma \ne \mu, |\bar{\beta}^\sigma \cap \beta^\mu| = 2\}\ .$$

Proof. To afford $G(35,18,9,9)$ we require a design D with parameters $(5,30,18,3,9)$, which we may take to be three identical copies of K_3^5.

Observe that K_3^5 has intersection distribution $\{0,3,6,0\}$ for each block (quasi-symmetric), so that the rule is well defined. Let B denote the 5×10 object-block incidence matrix of K_3^5 and let $Y = B^T B$. Let S be the 10×10 matrix that is 1 wherever Y is 1, 0 otherwise. Then

$$(5.6) \qquad Y = 3I + S + 2(J-S-I) = I - S + 2J .$$

The block adjacency rule states that C, the block adjacency matrix is given by

$$(5.7) \qquad C = \begin{array}{c c} & \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\ \begin{array}{c} D_1 \\ D_2 \\ D_3 \end{array} & \begin{bmatrix} S & J-S-I & J-S-I \\ J-S-I & S & J-S-I \\ J-S-I & J-S-I & S \end{bmatrix} \end{array} .$$

By Theorem 4.13 and its proof it suffices to show that

$$(5.8) \qquad \begin{cases} \text{(i)} & [B : B : B]C = 9J_{5\times 30}, \\ \text{(ii)} & C^2 = 9I + 9J - \begin{bmatrix} Y & Y & Y \\ Y & Y & Y \\ Y & Y & Y \end{bmatrix}. \end{cases}$$

Note first that

$$BJ_{5\times 10} = 6J_{5\times 10}, \quad J_{5\times 5}B = 3J_{5\times 5} .$$

Hence

$$BS = B(I+2J-Y) = B + 12J_{5\times 10} - BB^T B .$$

Using (1.12) we have

$$BS = B + 12J_{5\times 10} - (3I+3J_{5\times 5})B ,$$

or

$$BS = B + 12J_{5\times 10} - 3B - 9J_{5\times 10} = 3J_{5\times 10} - 2B .$$

Thus

$$[B:B:B]C = [-BS + 2BJ_{10\times 10} - 2B: -BS + 2BJ_{10\times 10} - 2B:-BS + 2BJ_{10\times 10} - 2B]$$

or

$$[B:B:B]C = [9J_{5\times 10}:9J_{5\times 10}:9J_{5\times 10}] ,$$

establishing (5.8) (i).

As for (5.8) (ii) we note that

$$(5.9) \qquad SJ = JS = 3J ,$$

while

$$(S^2)_{ij} = \begin{cases} 3 & \text{if} \quad i = j, \\ 1 & \text{if} \quad S_{ij} = 0, \quad i \neq j, \\ 0 & \text{if} \quad S_{ij} = 1, \end{cases}$$

so

$$(5.10) \qquad S^2 = J + 2I - S .$$

Setting $T = J - S - I$ we have

$$C^2 = \begin{bmatrix} S^2 + 2T^2 & 2ST + T^2 & 2ST + T^2 \\ 2ST + T^2 & S^2 + 2T^2 & 2ST + T^2 \\ 2ST + T^2 & 2ST + T^2 & S^2 + 2T^2 \end{bmatrix},$$

which reduces via (5.10), (5.9), and (5.6) to

$$C^2 = \begin{bmatrix} 9I & & 0 \\ 0 & 9I & 0 \\ 0 & 0 & 9I \end{bmatrix} + 9\begin{bmatrix} J & J & J \\ J & J & J \\ J & J & J \end{bmatrix} - \begin{bmatrix} Y & Y & Y \\ Y & Y & Y \\ Y & Y & Y \end{bmatrix}$$

so that (5.8) (ii) is in force. Hence the construction. □

We present one final construction using similar techniques. It is included to introduce a design that will afford two graphs in the present situation and will afford yet another construction in Section 6.

A (design-free) construction of a graph $G = G(36,15,6,6)$ is cited in Biggs (1971). We prove:

THEOREM 5.11. *The graph* $G = G(36,15,6,6)$ *is afforded by three copies* $\bar{D}_1, \bar{D}_2, \bar{D}_3$ *of the unique design* $\bar{D} = \bar{D}(6,10,5,3,2)$ *(see (4.15)) by means of the rule: if* β^μ *is a block of* $\bar{D}_\mu$ *then*

$$\Gamma_{G_B}(\beta^\mu) = \{\bar{\beta}^\mu \in \bar{D}_\mu : |\bar{\beta}^\mu \cap \beta^\mu| = 1\}$$
$$\bigcup \{\bar{\beta}^\sigma \in \bar{D}_\sigma : \mu \neq \sigma \ |\bar{\beta}^\sigma \cap \beta^\mu| = 2\} .$$

Proof. A design with parameters (6,30,15,3,6) is required; note that $\bar{D}$ has intersection distribution {0,6,3,0} for each block, so $\bar{D}$ is quasi-symmetric. Carry out the exact same procedure as in the proof of (5.5).

We note that both K_3^5 and $\bar{D}$ give rise to a familiar strongly regular graph.

COROLLARY 5.12. *There are three disjoint copies of the Petersen graph in the graph* G(35,18,9,9) *constructed in* (5.5), *while the graph* G(36,15,6,6) *constructed in* (5.11) *contains three disjoint copies of the complement of the Petersen graph.*

Proof. In the proof for G(35,18,9,9), S satisfied (5.9) and (5.10), making it (see (1.4) and (1.5)) the adjacency matrix of a strongly regular graph with parameters (10,3,0,1). The three Petersen graphs arise from the three copies of S used in (5.7). For G(36,15,6,6) the 10×10 matrix $\bar{S}$ whose rows and columns are indexed by the blocks of $\bar{D}$ and for which

$$\bar{s}_{ij} = \begin{cases} 1 & \text{if } |\beta_i \cap \beta_j| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

is seen to satisfy $\bar{S}^2 = 2I - \bar{S} + 4J$ and $\bar{S}J = 6J$. Thus $\bar{S}$ is (via (1.4) and (1.5)) the adjacency matrix of a strongly regular graph whose parameters are (10,6,3,4), the complement of the Petersen graph. The three copies of this graph arise from the three copies of $\bar{D}$ used in Theorem (5.11). □

6. *The Dodecahedron.*

This chapter is devoted to the design construction of a strongly regular graph with parameters (26,10,3,4).

First we give the design construction for such a graph and establish a close tie between the graph and the Petersen graph via a "fusion" of blocks. Second we are able to exploit the presence of the Petersen graph to determine the automorphism group of the graph and give it a pleasing geometric realization.

By Theorem 2.3 the design construction of a strongly regular graph G = G(26,10,3,4) requires a design D with parameters (6,20,10,3,4). There are two natural choices for D, namely,

(6.1) (i) $D = K_3^6$,

(ii) D is obtained from two copies of the unique (see Hall (1967)) $\bar{D} = \bar{D}(6,10,5,3,2)$. Here we write $D = \bar{D}^2$.

For reasons discussed in Thompson (1979), we work with the design of (6.1) (ii) and obtain

THEOREM 6.2. *The design* $\bar{D}^2$ *affords a strongly regular graph* $G_{\bar{D}^2}(26,10,3,4)$.

Proof. We will refer to the two copies of $\bar{D}$ comprising $\bar{D}^2$ as $\bar{D}_0$ and $\bar{D}_1$, the subscripts taken modulo 2. Represent the blocks of $\bar{D}_\sigma$, $\sigma = 0,1$, as

$$\{(1,2,3)^\sigma,\ (1,2,5)^\sigma,\ (2,3,6)^\sigma,\ (2,4,6)^\sigma,\ (1,3,4)^\sigma,$$
$$(3,4,5)^\sigma,\ (2,4,5)^\sigma,\ (3,5,6)^\sigma,\ (1,4,6)^\sigma,\ (1,5,6)^\sigma\}.$$

Since we will be using two identical copies of $\bar{D}$, we note that there is no contradiction in agreeing that each block is adjacent to the block identical to it. This amounts to putting three triangles (from objects) on each edge $\langle\beta^0,\beta^1\rangle$. Having agreed to this convention, we cannot have a block $\beta^\sigma(\sigma = 0$ or $1)$ adjacent to two identical blocks (since otherwise there would be four triangles on the resulting edge). Hence $\Gamma_{G_B}(\beta^0) \cap \Gamma_{G_B}(\beta^1) = \emptyset$ for every $\beta^0 \in \bar{D}_0$. The rule that $\beta^0 \sim \beta^1$ for every $\beta^0 \in \bar{D}_0$ is almost sufficient to determine the block adjacency rule. Since $\Gamma_{G_B}(\beta^\sigma)$ must contain four repetitions of each of the objects not in β^σ and three repetitions of each of the objects in β^σ, (see Theorem 4.13 (ii)), the following rule determines the makeup of the blocks of $\Gamma_{G_B}(\beta^\sigma)$:

(6.3) For $\beta^\sigma \in \bar{D}^2$, where $\beta^\sigma = (i,j,k)^\sigma$, we take $\Gamma_{G_B}(\beta^\sigma)$ to consist of $\beta^{\sigma+1}$ plus the six uniquely determined (up to copy) blocks $(\ell,m,i)^{\sigma_1}$, $(\ell,m,j)^{\sigma_2}$, $(\ell,n,i)^{\sigma_3}$, $(\ell,n,k)^{\sigma_4}$, $(m,n,j)^{\sigma_5}$, $(m,n,k)^{\sigma_6}$, where ℓ, m, n are the three objects not in β^σ.

Here, at least, is a design for which the block adjacency rule can (up to copy) be uniquely prescribed, independent of the block. For example, with $\beta^0 = (1,2,3)^0$ we get

$$\Gamma_{G_B}(\beta^0) = \{\beta^1,(3,4,5)^{\sigma_1},(2,4,5)^{\sigma_2},(1,4,6)^{\sigma_3},\ (2,4,6)^{\sigma_4},(3,5,6)^{\sigma_5},(1,5,6)^{\sigma_6}\}.$$

The only undecided part that must be settled is the selection of $\sigma_1,\sigma_2,\ldots,\sigma_6$. This is where Theorem 4.13 (iii) is used. Note that (6.3) requires

$$(6.4) \qquad \beta^\sigma \sim \beta_i^\tau \text{ if and only if } \beta^{\sigma+1} \sim \beta_i^{\tau+1}$$

which cuts the work in half. Also, (6.4) is equivalent to saying that the map (involution) $\theta:\beta^\sigma \to \beta^{\sigma+1}$ defined on $\bar{D}^2$ is an automorphism of G. We may write the block intersection matrix Y of $\bar{D}^2$ as

$$(6.5) \qquad Y = \begin{bmatrix} \bar{Y} & \bar{Y} \\ \bar{Y} & \bar{Y} \end{bmatrix},$$

where $\bar{Y}$ is the block intersection matrix of $\bar{D}$, and C may therefore be written as

$$(6.6) \qquad C = \begin{bmatrix} C_1 & C_2 \\ C_2 & C_1 \end{bmatrix},$$

since $\Gamma_{G_B}(\beta^{\sigma+1})$ is determined by $\Gamma_{G_B}(\beta^\sigma)$. We also have that $C_1 \circ C_2 = 0$. There is enough information now to limit significantly the choices in the makeup of $\Gamma_{G_B}(\beta^\sigma)$.

To begin the construction we take $\Gamma_{G_B}((1,2,3)^0) = \{(1,2,3)^1, (1,4,6)^1, (1,5,6)^1, (2,4,5)^1, (2,4,6)^0, (3,5,6)^0, (3,4,5)^1\}$ without loss of generality. This also determines $\Gamma_{G_B}((1,2,3)^1)$ because of (6.4). This choice yields Figure 5.

Consider $\Gamma_{G_B}((1,4,6)^1)$. It must consist of $(1,4,6)^0$ and $(1,2,3)^0$ and, up to copy choice, $(1,2,5)^{\sigma_1}$, $(4,3,5)^{\sigma_2}$, $(4,2,5)^{\sigma_2}$, $(6,3,5)^{\sigma_4}$, $(6,2,3)^{\sigma_5}$. From Theorem 4.13 (iii) applied to the edge $\langle(1,2,3)^0, (1,4,6)^1\rangle$ we see that $(1,4,6)^1$ is adjacent to two of the blocks of $\Gamma_{G_B}((1,2,3)^0) \setminus (1,2,3)^1$, i.e., two of $(2,4,5)^1$, $(3,5,6)^0$, $(3,4,5)^1$. Applying Theorem 4.13 (iii) to $(1,2,3)^0$ and to some block of $\Gamma_{G_B}((1,2,3)^1)$ we see that an additional path of length two is needed between $(1,2,3)^0$ and each of the blocks of $\Gamma_{G_B}((1,2,3)^1)$. Hence

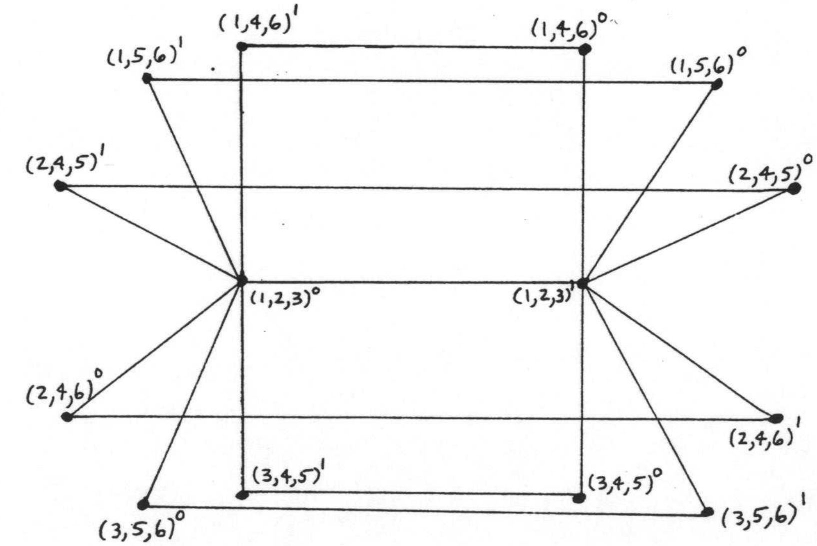

Figure 5. $\Gamma_{G_B}((1,2,3)^\sigma)$, $\sigma = 0,1$.

$(1,4,6)^1$, for example, is adjacent to one of $(2,4,5)^0$, $(3,5,6)^1$, or $(3,4,5)^0$. There are analogous requirements for $\Gamma_{G_B}(\beta)$ for each

$\beta^{\tau} \in \Gamma_{G_B}((1,2,3)^0)\backslash(1,2,3)^1$. In the interest of symmetry we choose the two blocks of $\Gamma_{G_B}((1,2,3)^0)\backslash(1,2,3)^1$ to be connected with $(1,4,6)^1$ as that pair having but one common object. Proceeding in this manner we obtain:

$$
(6.7)\qquad \left\{\begin{array}{l}
\Gamma_{G_B}((1,4,6)^1) \supseteq \{(2,4,5)^1,\ (3,5,6)^0,\ (3,4,5)^0\},\\
\Gamma_{G_B}((1,5,6)^1) \supseteq \{(2,4,6)^0,\ (3,4,5)^1,\ (2,4,5)^0\},\\
\Gamma_{G_B}((2,4,5)^1) \supseteq \{(1,4,6)^1,\ (3,5,6)^0,\ (1,5,6)^0\},\\
\Gamma_{G_B}((2,4,6)^0) \supseteq \{(1,5,6)^1,\ (3,4,5)^1,\ (3,5,6)^1\},\\
\Gamma_{G_B}((3,5,6)^0) \supseteq \{(1,4,6)^1,\ (2,4,5)^1,\ (2,4,6)^1\},\\
\Gamma_{G_B}((3,4,5)^1) \supseteq \{(1,5,6)^1,\ (2,4,6)^0,\ (1,4,6)^0\}.
\end{array}\right.
$$

We have now nearly completed fourteen rows of C. Applying Theorem 4.3 (i) repeatedly to these rows and the remaining rows of C, and keeping in mind the rules (6.3) and (6.4) we eventually obtain the block adjacency matrix listed in (6.8). The graph obtained is unique up to the switching of labels on any pair of identical blocks.

Taking the 6×10 object-block incidence matrix $\bar{B}$ of $\bar{D}'$ to get $B = [\bar{B}\ \bar{B}]$ for the object-block incidence matrix of $\bar{D}^2$ we observe that

$$
A = \begin{bmatrix} 0 & B \\ B^T & C \end{bmatrix}
$$

satisfies (1.5) and (1.6). Hence we have a design construction for a strongly regular graph, which we will denote by $G_{\bar{D}^2} = G_{\bar{D}^2}(26,10,3,4)$. $\square$

It would be gratifying to have a more pleasing picture of $G_{\bar{D}^2}$ than its adjacency matrix. To that end we introduce the next definition.

Definition 6.9. Let $D = D(v,2b,2r,k,2\lambda)$ be a design obtained from two copies D_0' and D_1' of $D' = D'(v,b,r,k,\lambda)$, the subscripts taken modulo 2. Label the blocks of D as β^{σ} with $\sigma = 0,1$ denoting respective copy. Let D afford $G = G(n,2r,c,2\lambda)$. Let $\theta:\beta^{\sigma} \rightarrow \beta^{\sigma+1}$ be a block graph automorphism and let $\beta_i^0 \sim \beta_i^1$ for all i. Define the *fusion subgraph* G_f of G_B as that graph whose vertices β are identified as block pairs

$$
C=\begin{array}{r|cccccccccccccccccccc|}
 & (1,2,3)^0 & (1,2,5)^0 & (2,3,6)^0 & (2,4,6)^0 & (1,3,4)^0 & (3,4,5)^0 & (2,4,5)^0 & (3,5,6)^0 & (1,4,6)^0 & (1,5,6)^0 & (1,2,3)^1 & (1,2,5)^1 & (2,3,6)^1 & (2,4,6)^1 & (1,3,4)^1 & (3,4,5)^1 & (2,4,5)^1 & (3,5,6)^1 & (1,4,6)^1 & (1,5,6)^1 \\
\hline
(1,2,3)^0 & 0&0&0&1&0&0&0&1&0&0&1&0&0&0&0&1&1&0&1&1 \\
(1,2,5)^0 & 0&0&0&1&0&1&0&0&0&0&0&1&1&0&1&0&0&1&1&0 \\
(2,3,6)^0 & 0&0&0&0&0&0&0&0&1&1&0&1&1&0&1&1&1&0&0&0 \\
(2,4,6)^0 & 1&1&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&1&0&1 \\
(1,3,4)^0 & 0&0&0&0&0&0&1&1&0&0&0&1&1&1&1&0&0&0&0&1 \\
(3,4,5)^0 & 0&1&0&0&0&0&0&0&0&1&1&0&1&1&0&1&0&0&1&0 \\
(2,4,5)^0 & 0&0&0&0&1&0&0&0&1&0&1&0&1&0&0&0&1&1&0&1 \\
(3,5,6)^0 & 1&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&1&1&1&0 \\
(1,4,6)^0 & 0&0&1&0&0&0&1&0&0&0&1&1&0&0&0&1&0&1&1&0 \\
(1,5,6)^0 & 0&0&1&0&0&1&0&0&0&0&1&0&0&1&1&0&1&0&0&1 \\
(1,2,3)^1 & 1&0&0&0&0&1&1&0&1&1&0&0&0&1&0&0&0&1&0&0 \\
(1,2,5)^1 & 0&1&1&0&1&0&0&1&1&0&0&0&0&1&0&1&0&0&0&0 \\
(2,3,6)^1 & 0&1&1&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&1&1 \\
(2,4,6)^1 & 0&0&0&1&1&1&0&1&0&1&1&1&0&0&0&0&0&0&0&0 \\
(1,3,4)^1 & 0&1&1&1&1&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0 \\
(3,4,5)^1 & 1&0&1&1&0&1&0&0&1&0&0&1&0&0&0&0&0&0&0&1 \\
(2,4,5)^1 & 1&0&1&0&0&0&1&1&0&1&0&0&0&0&1&0&0&0&1&0 \\
(3,5,6)^1 & 0&1&0&1&0&0&1&1&1&0&1&0&0&0&1&0&0&0&0&0 \\
(1,4,6)^1 & 1&1&0&0&0&1&0&1&1&0&0&0&1&0&0&0&1&0&0&0 \\
(1,5,6)^1 & 1&0&0&1&1&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0
\end{array}
$$

(6.8)

$\beta = (\beta^0, \beta^1)$; β^0, β^1 in G_B. The rule of adjacency in G_f is

$$\Gamma_{G_f}(\beta) = \{\bar\beta = (\bar\beta^0, \bar\beta^1) \in G_f : \beta^\sigma \sim \bar\beta^\tau, \sigma = 0 \text{ or } 1, \ \tau = 0 \text{ or } 1\} .$$

THEOREM 6.10. *Let* (G,D) *be as described in Definition 6.9. The block adjacency matrix* C *of* G_B *may be written*

$$C = \begin{bmatrix} C_1 & C_2 \\ C_2 & C_1 \end{bmatrix}$$

and the adjacency matrix of G_f *is* $C_1 + C_2 - I - C_1 \circ C_2$. *In particular, for* $c = k$ *the adjacency matrix is* $C_1 + C_2 - I$ *and* G_f *has valence* $2r - k - 1$.

Proof. Requiring that $\theta : \beta^\sigma \to \beta^{\sigma+1}$ be a G_B automorphism is equivalent to imposing the rule $\beta^\sigma \sim \beta_i^\tau$ if and only if $\beta^{\sigma+1} \sim \beta_i^{\tau+1}$, so clearly C has the form indicated. From the presence of θ we note that $\beta^0, \beta^1 \sim \bar\beta^\sigma$ entails $\beta^0, \beta^1 \sim \bar\beta^{\sigma+1}$. Thus, if $\Gamma_{G_B}(\beta_i^0) \cap \Gamma_{G_B}(\beta_i^1) = \{\beta_{i_1}^0, \beta_{i_1}^1, \ldots, \beta_{i_p}^0, \beta_{i_p}^1\}$, say, then $(C_1 + C_2)_{ii_m} = 2$, $1 \le m \le p$, and so $(C_1 \circ C_2)_{ii_m} = 1$, $1 \le m \le p$. Note also that C_2 has $(C_2)_{ii} = 1$, $1 \le i \le b$. Thus G_f has the indicated adjacency matrix.

In the case $c = k$ then every edge $\langle\beta^0, \beta^1\rangle$ of G_B has no triangles, whence $C_1 \circ C_2 = 0$. Thus $C_1 + C_2 - I$ is the correct G_f adjacency matrix in this case. Clearly, the valence in G_f is $2r-k-1$ in the case $c = k$. □

We now establish the relationship between $G_{\bar D^2}$ and the Petersen graph

THEOREM 6.11. *For* $G_{\bar D^2}$, *the fusion subgraph* G_f *is the complement of the Petersen graph having parameters* $(10,6,3,4)$.

Proof. We have Definition 6.9 in force and $c = k$. From (6.8) we verify that $C_1 + C_2 - I = C_f$, the adjacency matrix of G_f, satisfies $C_f J = 6J$ and $C_f^2 = 2I - C_f + 4J$. Hence, by (1.4) and (1.5), G_f is strongly regular with parameters $(10,6,3,4)$ and hence is the complement of the Petersen graph. □

From the fusion subgraph G_f obtained above we can realize $G_{\bar D^2}$ geometrically. To motivate the next theorem we make the following observation:

PROPOSITION 6.12. *A Petersen graph can be obtained from the ten antipodal pairs of vertices of a dodecahedron.*

Proof. Recall that a dodecahedron has twelve pentagonal faces, twenty vertices, thirty edges. Identify an antipodal pair of vertices as one fused vertex and construct a graph on the ten fused vertices by adjoining two fused vertices if and only if the corresponding original four dodecahedral vertices are the vertices on two dodecahedral edges. Clearly the result is a Petersen graph. □

THEOREM 6.13. $G_{\bar D^2}$ *may be constructed from the faces and vertices of a dodecahedron.*

Proof. The six objects of the design $\bar D$ are the six antipodal pairs of pentagons on the dodecahedron. Label the (fused) faces $\{1,2,3,4,5,6\}$. The twenty blocks of the multiple design are the twenty vertices of the dodecahedron where each block contains those objects which, as pentagons, meet at the given block vertex. (See Figure 6.)

We see that a copy of the ten blocks of $\bar D$ is obtained by taking the five blocks forming any pentagon and the five further blocks that are dodecahedrally adjacent to these (much like a "top half" of the dodecahedron). The other copy of $\bar D$ is made up of the remaining ten corners (antipodal to the first ten). Note that antipodal corners

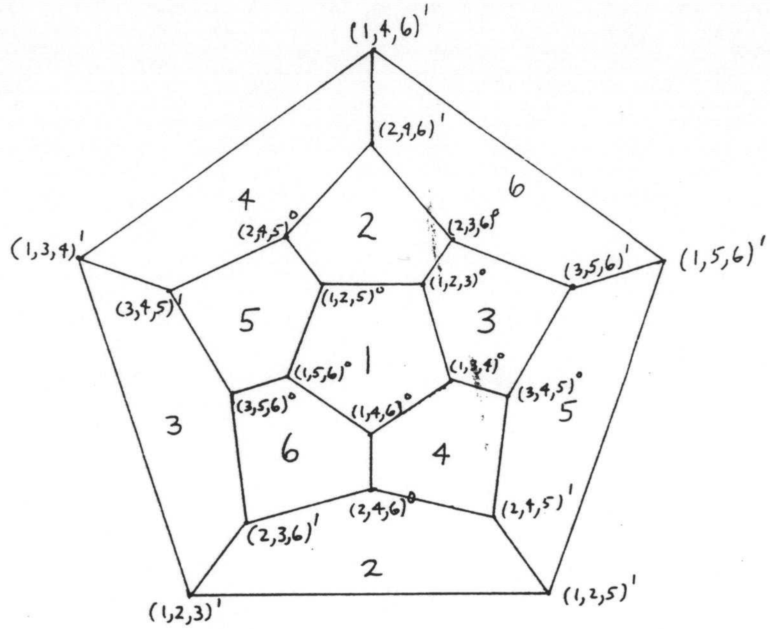

Figure 6. The Dodecahedron, Embedded in the Plane.

(Object labeling done so as to be consistent with $\bar{D}$.)

correspond to identical blocks. Our design construction of $G_{\bar{D}^2}$ is then obtained from the dodecahedron as follows:

Taking the six objects (antipodal pentagon pairs) and twenty blocks for the twenty-six vertices of $G_{\bar{D}^2}$, we connect an object vertex to the ten blocks (dodecahedral corners) bordering the corresponding pentagonal pair. We connect a block $\beta^{\sigma+1}$ antipodal to β^{σ}, and to those six blocks which, as dodecahedral corners, are at distance three from β^{σ} on the dodecahedron. This rule, in fact, gives the adjacency matrix A as described in Theorem 6.2. □

By means of the dodecahedral realization we have the next theorem.

THEOREM 6.14.

(i) $\mathrm{Aut}(G_{\bar{D}^2}) \cong I$, *the icosahedral group;*

(ii) $\mathrm{Aut}(\bar{D}) \cong A_5$*;*

(iii) $\mathrm{Aut}(\bar{D}^2) \cong A_5 \times (Z_2)^{10}$*;*

(iv) The map that fuses $(G_{\bar{D}^2})_B$ *into* G_f *induces a homomorphism* $\phi : I \to \mathrm{Aut}(P)$, *where* $\ker(\phi) \cong Z_2$ *and* $\mathrm{Aut}(P)$ *is the automorphism group of the Petersen graph; hence* $A_5 < \mathrm{Aut}(P)$.

Proof. It is well known (Benson and Grove (1971)) that the group of symmetries of a regular dodecahedron is $I \cong A_5 \times Z_2$, A_5 being the rotation subgroup and Z_2 the subgroup that interchanges antipodal dodecahedral vertices. Thus we have (i).

For (ii) we may identify antipodal dodecahedral vertices as the *same* block. From our embedding of $\bar{D}$ in a dodecahedron we have that $\mathrm{Aut}(\bar{D})$ consists of precisely those object permutations (pentagon permutations) that preserve the dodecahedron. Hence $\mathrm{Aut}(\bar{D})$ is isomorphic to the rotation subgroup of I, i.e., $\mathrm{Aut}(\bar{D}) \cong A_5$.

For (iii) observe that the switching of the labels on any pair (β^0, β^1) of blocks in $\bar{D}^2$ preserves $\bar{D}^2$ so (iii) follows.

For (iv) we note that for a permutation of the vertices (blocks) resulting from a rotation $\psi \in A_5$ of the dodecahedron we have $\psi(\beta_i^{\sigma}) = \beta_j^{\tau}$ if and only if $\psi(\beta_i^{\sigma+1}) = \beta_j^{\tau+1}$. Thus any rotation may be represented by a permutation matrix of the form

$$\begin{bmatrix} R_1 & R_2 \\ R_2 & R_1 \end{bmatrix},$$

and hence any matrix Q representing an element of I is of the form

$$Q = \begin{bmatrix} R_1 & R_2 \\ R_2 & R_1 \end{bmatrix} \text{ or } Q = \begin{bmatrix} R_1 & R_2 \\ R_2 & R_1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} R_2 & R_1 \\ R_1 & R_2 \end{bmatrix}.$$

Define $\phi : I \to \mathrm{Aut}(P)$ by setting $\phi(Q) = (R_1 + R_2)$. Clearly ϕ is a homomorphism for which

$$\phi(Q)(C_1 + C_2 - I) = (C_1 + C_2 - I)\phi(Q)$$

if and only if $QC = CQ$. Also,

$$\ker(\phi) = \left[\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, (I)\right].$$

Therefore $I/\ker(\phi) \cong A_5 < \mathrm{Aut}(P)$. □

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Pepperdine University
Malibu
California 90265

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